# DIRECTION OF ARRIVAL ESTIMATION BASED ON INFORMATION GEOMETRY

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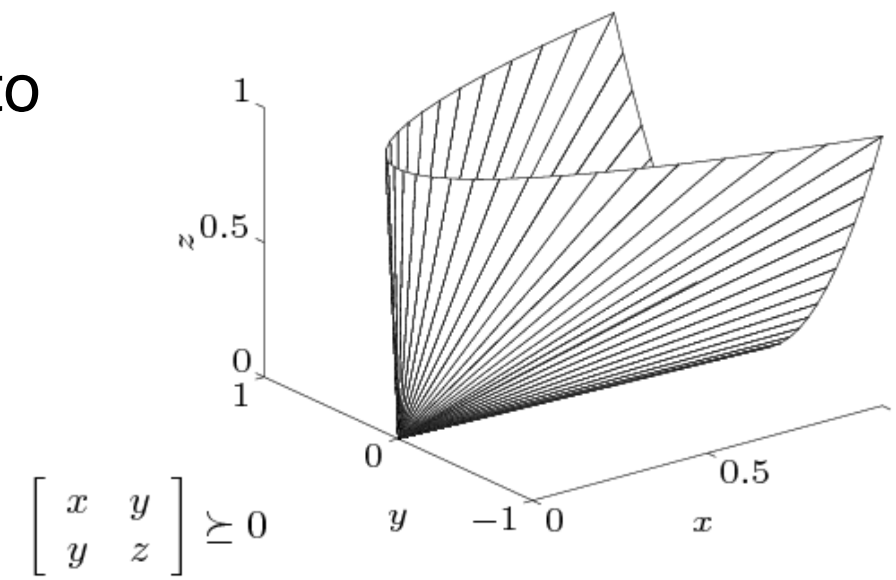
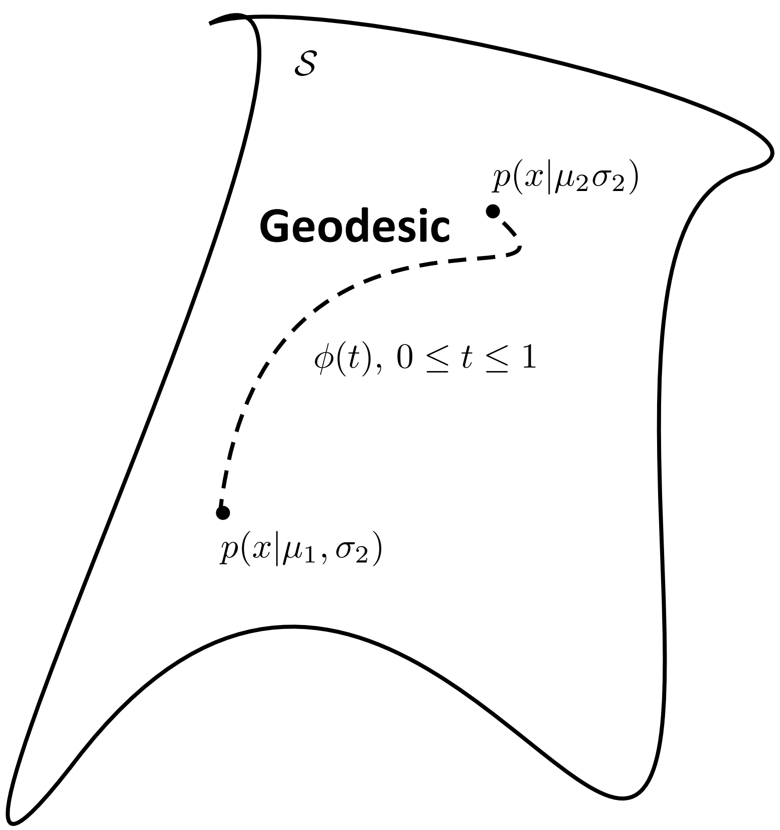
# Motivation

- The space of covariance matrices is **not** an **Euclidean space**
  - The set of positive definite matrices is a **convex cone**

- Measure of closeness must be modified in order to adapt to the geometry of the space
  [Amari, 1997] [Cencov, 2000]




[Vandenberghe, 2012]



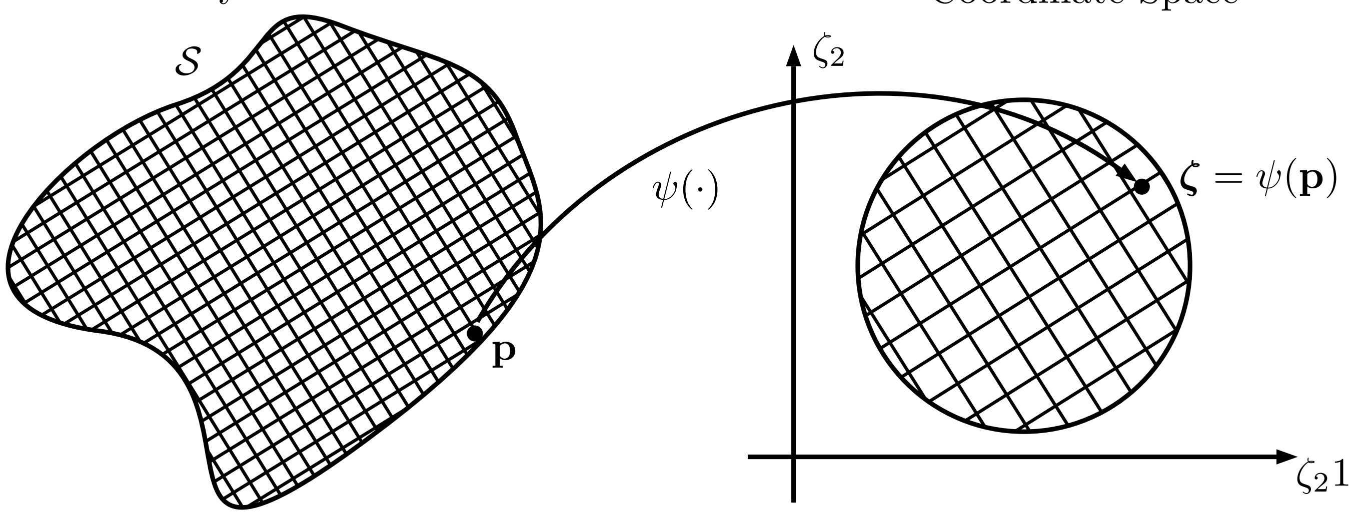

Manifold of Probability Distributions

- Gaussian normal probability distributions define a manifold with known geometry. [Lang, 2001]

- In this manifold, distances can be measured based on the **Fisher's information matrix** [Rao, 1945]
  - Fundamental results in estimation theory [Crámer,1946]

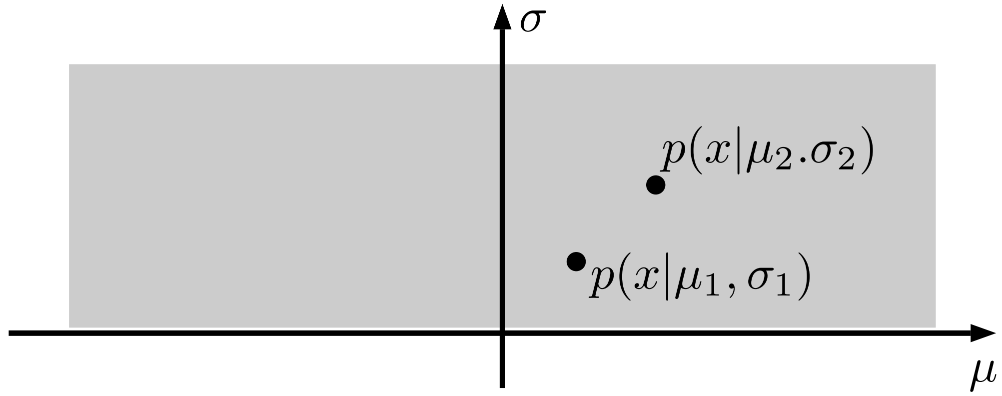
# Information Geometry

- Information geometry considers probability distributions as structures of **differential geometry**
  - **Not necessary flat spaces anymore** [Rao, 1945]




Mapping $\psi(\cdot)$ between Manifold and Coordinate Space

- Probability distributions can be treated as points in the space (**manifold**) [Amari, 1980] [Amari, 1997]




Parameter Space $(\mu, \sigma)$

**Where** to measure the distances**?**

**How** to measure the distances**?**

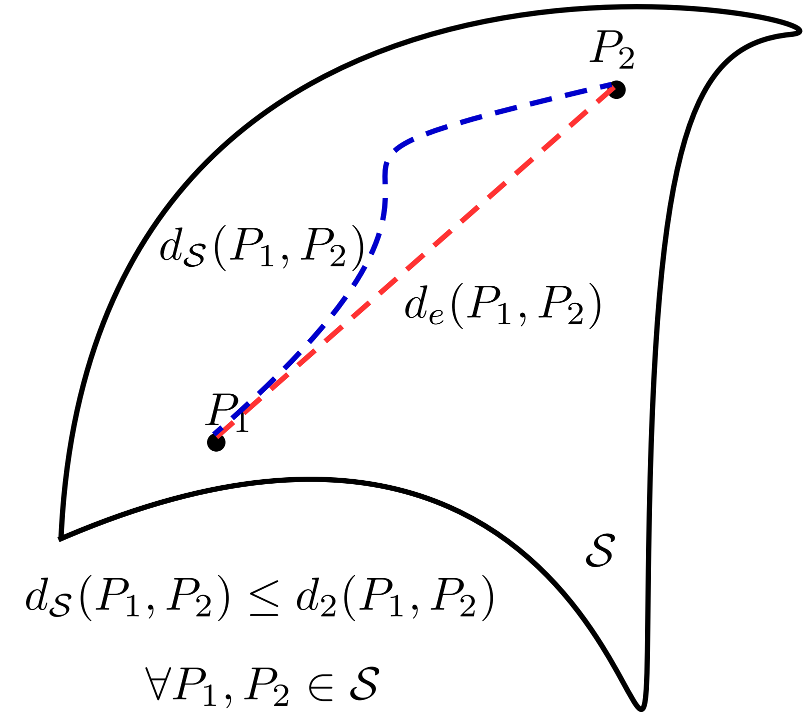
# Information Geometry

- **Where** to measure?
  - Naturally, in the **manifold of probability distributions!**

- Then**, how** we should measure distances?
  - **Space is not necessary flat! (Euclidean distance is not adequate anymore!)**
  - The Fisher's information matrix is introduced in [Rao,1945] as **Riemannian metric** for the manifold

$$\mathbf{G}_{i,j}(\boldsymbol{\theta}) = -E\{\frac{\partial^2 \ln p(\mathbf{x}|\boldsymbol{\theta}_i)}{\partial \boldsymbol{\theta}_i \partial \boldsymbol{\theta}_j}\}$$

  - The straight line's concept is extended in manifolds by **geodesic curves**




**Geodesic** : Parametrized curve which minimize

$$L_{\boldsymbol{\theta}} = \int_{t_1}^{t_2} \|\frac{d\boldsymbol{\theta}(t)}{dt}\|_{\mathcal{S}} = \int_{t_1}^{t_2} \sqrt{\frac{d\boldsymbol{\theta}(t)^T}{dt}\mathbf{G}(\boldsymbol{\theta}(t))\frac{d\boldsymbol{\theta}(t)}{dt}}$$

[Lang, 2001]

**Induced norm**

$$\|\mathbf{x}\|_{\mathcal{S}} = \sqrt{\mathbf{x}^T\mathbf{G}\mathbf{x}}$$

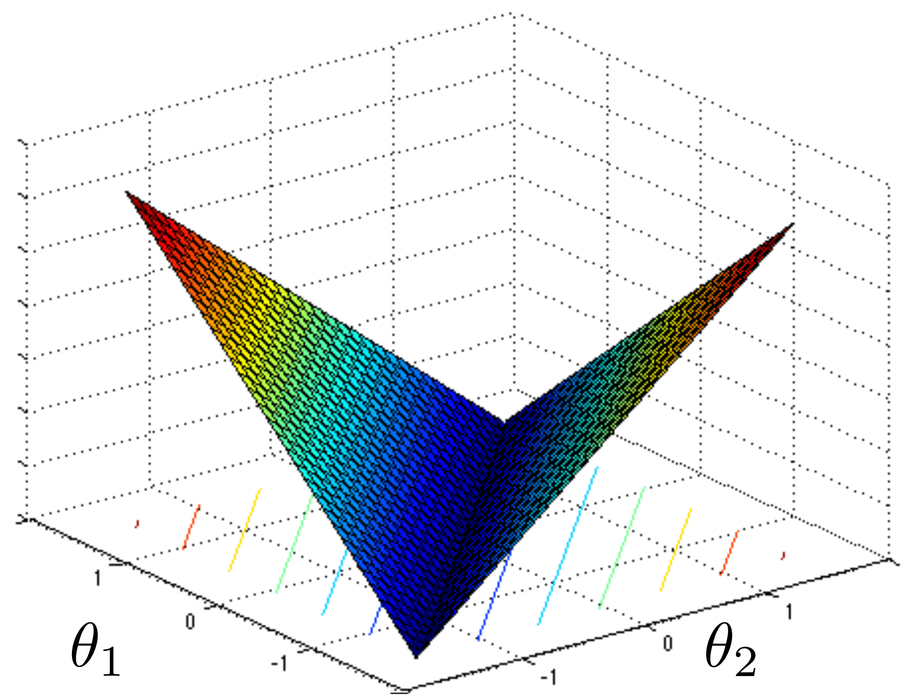
vectors in the *tangent* space of $\mathcal{S}$

# Why is this metric relevant?

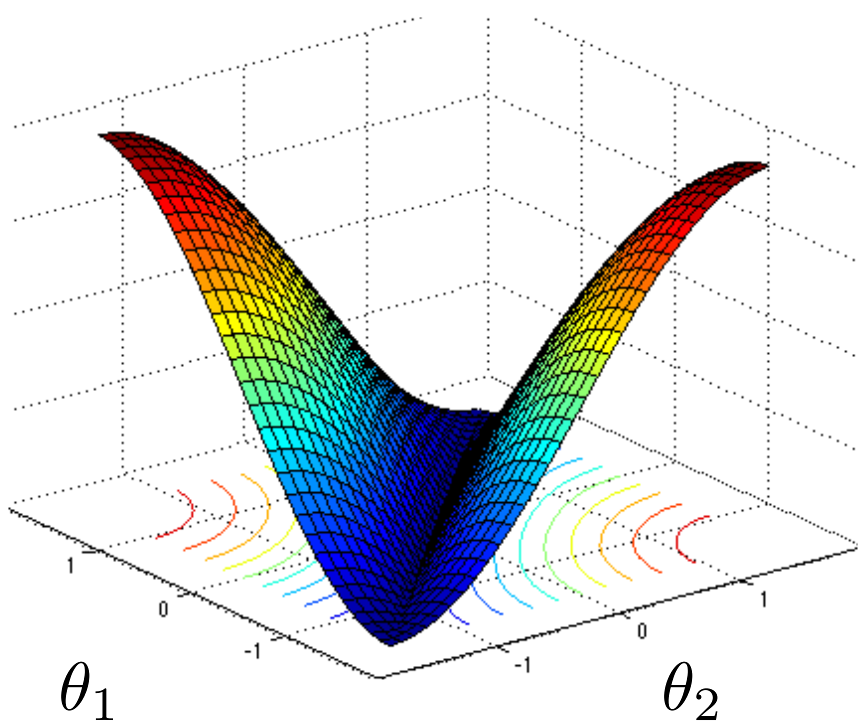
**Crámer-Rao Lower Bound**

$$Var(\hat{\boldsymbol{\theta}}) \geq CRLB(\boldsymbol{\theta}) = \mathbf{G}^{-1}(\boldsymbol{\theta})$$

where $\mathbf{G}(\boldsymbol{\theta})$ is the Fisher information matrix.



Euclidean similarity matrix between pairs of Azimuth angles in 1D Array

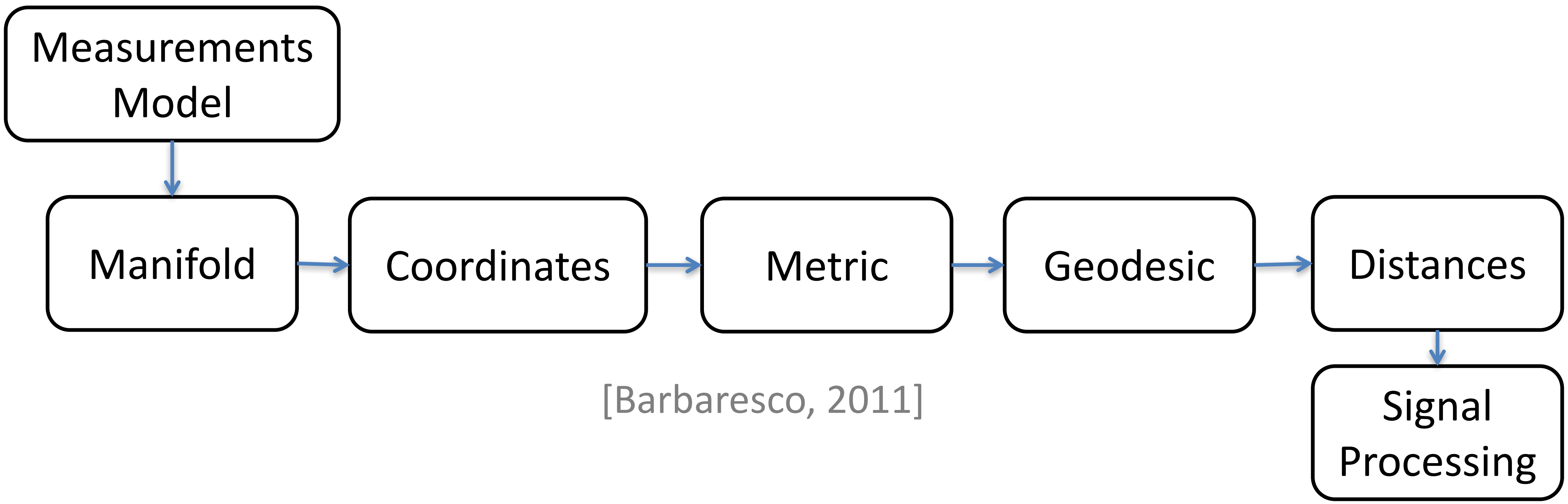


Information Geometry similarity matrix between pairs of Azimuth angles in 1D Array

- **Benefits of the metric:**
  - Geometrically correct (**manifold distance**)
  - Contains information about estimation capabilities (**Fisher information**)

# Information Geometry of Covariance Matrices

- **Information geometry based processing**




[Barbaresco, 2011]

**Measurements model**

$$\mathbf{x} \sim \mathcal{CN}(\mathbf{0}, \mathbf{R}(\boldsymbol{\theta}))$$

**Fisher's Information**

$$\mathbf{G}_{i,j}(\boldsymbol{\theta}) = -E\{\frac{\partial^2 \ln p(\mathbf{x}|\boldsymbol{\theta}_i)}{\partial \boldsymbol{\theta}_i \partial \boldsymbol{\theta}_j}\}$$

**Distance between Multivariate Gaussian distributions**

$$d(p(\mathbf{x}|\mathbf{R}_1), p(\mathbf{x}|\mathbf{R}_2)) \triangleq d(\mathbf{R}_1, \mathbf{R}_2) = \sum_{i=1}^{n} (\log a_i)^2$$

where $a_1, \ldots, a_n$ are the roots of $det(\lambda \mathbf{R}_1 - \mathbf{R}_2)$ .

# IG Based DOA Estimation

- **Data Model**

$$\mathbf{x}[k] = \sum_{i=1}^{D} \mathbf{a}(\theta_i) s_i[k] + \mathbf{n}[k] = \mathbf{A}\mathbf{s}[k] + \mathbf{n}[k]$$

$$\mathbf{a}(\theta_i) = [1, \psi_i, \ldots, \psi_i^{M-1}] \qquad \psi_i = \exp(j2\pi \frac{l}{\lambda} \sin(\theta_i))$$

- Classical signal processing
  - **Flat metric, Normed Space, Not optimal**

- **Information Geometry of Covariance matrices**
  - Takes into account the structure of covariance matrices' manifold

    [S.T. Smith, 2000] [Barbaresco, 2011] [B. Balaji, 2014] [Ke Sun, 2014]

**Problem: Estimation of DOA by minimization of information geometry distance**

**Multivariate Optimization Problem**

$$\min_{\tilde{\mathbf{R}}, \tilde{\mathbf{A}} \in \mathcal{A}} \quad d(\mathbf{R}_{xx}, \tilde{\mathbf{R}})$$

$$s.t \quad \tilde{\mathbf{R}} = \tilde{\mathbf{A}}\tilde{\mathbf{A}}^H$$

$$\mathbf{R}_{xx} = E\{\mathbf{x}\mathbf{x}^H\}$$

$\mathcal{A}$ : Antenna array manifold

# IG Based DOA Estimation

- **Rank-1 Problem Instance**

Feasible set $$\mathcal{A}_1 = \{\mathbf{a}(\phi),\ \phi \in [-\pi/2, \pi/2]\}$$

- IG-based pseudo-spectrum **(Line search)**

$$f(\phi) = \frac{1}{d(\hat{\mathbf{R}}_{xx}, \tilde{\mathbf{R}}(\phi))}, \ \phi \in [-\pi/2, \pi/2]$$

where

$$\tilde{\mathbf{R}}(\phi) = \mathbf{a}(\phi)\mathbf{a}^H(\phi)$$

- **Relation to MVDR**

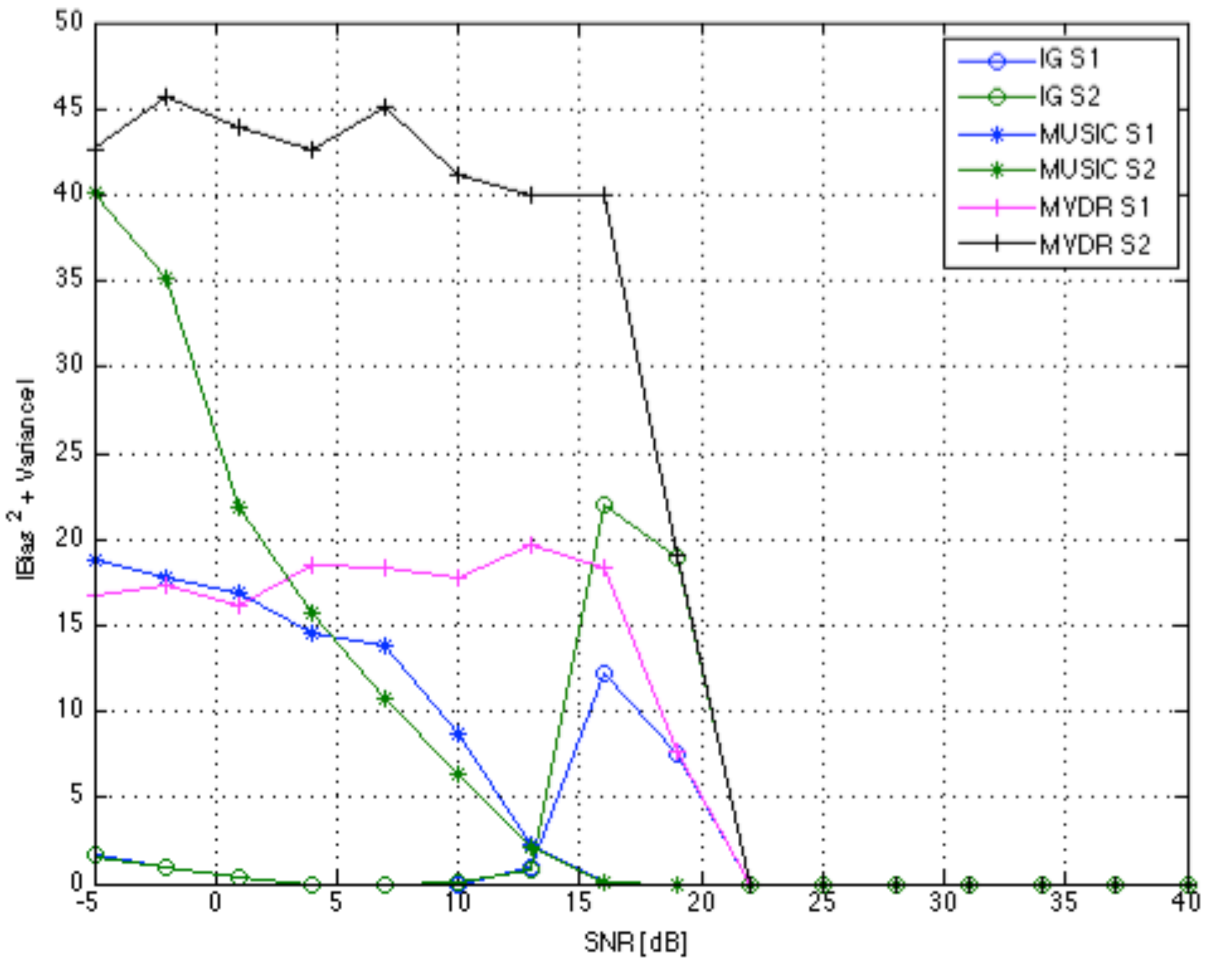
$$f(\phi) = \frac{1}{(\log \mathbf{a}(\phi)^H \hat{\mathbf{R}}_{xx}^{-1} \mathbf{a}(\phi))^2}, \ \phi \in [-\pi/2, \pi/2]$$

- *Differences* in performance for

$$\mathbf{a}^H(\phi)\hat{\mathbf{R}}_{xx}^{-1}\mathbf{a}(\phi) \lesssim 1$$

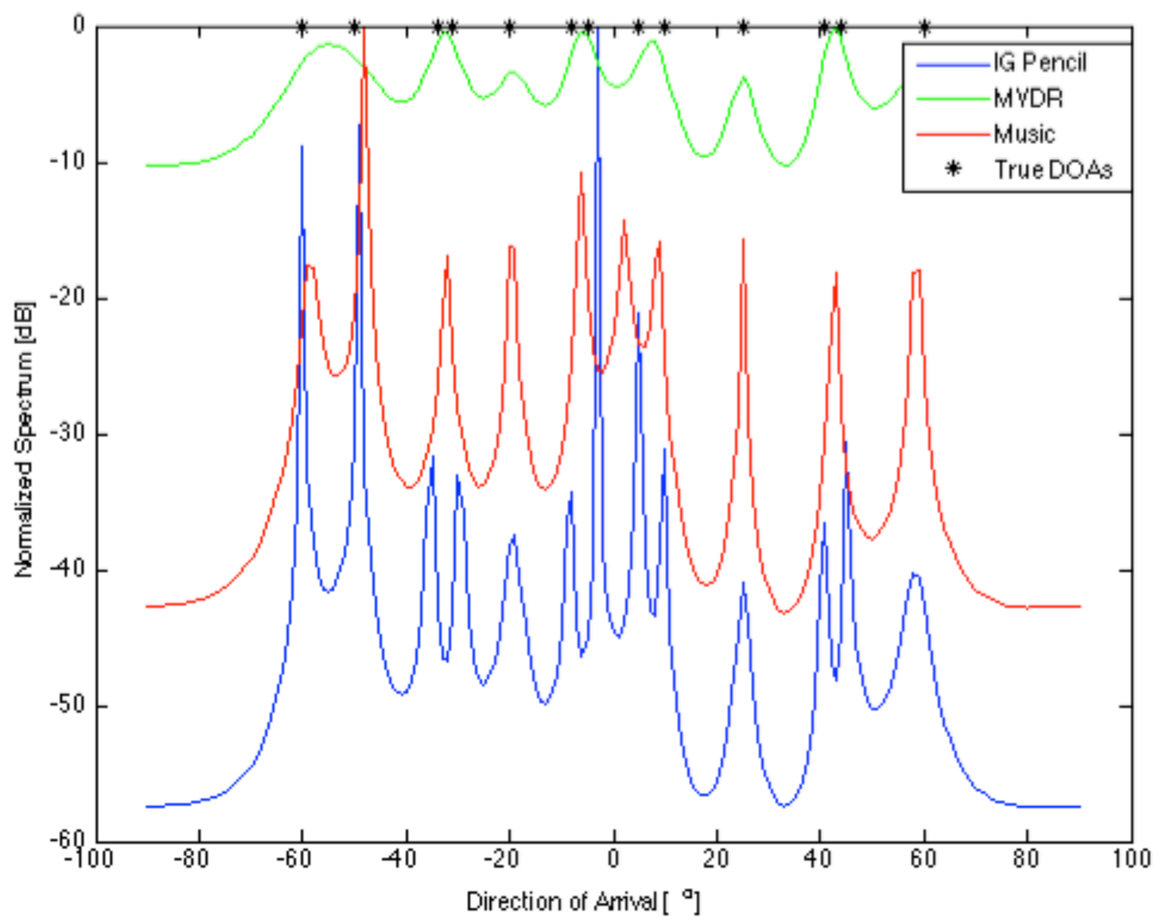
# Results

**Close sources**




Comparison for two sources at [-20º, -23º]

**More sources than antennas**




13 Sources present in data acquired by 11 antenna elements under a SNR of 10dB

# Conclusions

- By considering the geometry of the manifold, IG provides natural distances which could lead to improvements **in algorithms' performance.**

- We obtained a **simple DOA estimation algorithm** based on **IG geodesic distances**
  - MVDR is related to the **rank-1 instance** of the IG-based DOA estimation problem.

- IG-based DOA estimation leads to
  - **Improvements in resolution** power for closed sources at low SNR
  - **Identification** of **more sources** than antenna elements