Small Sample Size Performance of the Energy Detector

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***Abstract*—We examine the small sample size performance of the energy detector for spectrum sensing in AWGN. By making use of the cube-of-Gaussian approximation of chi-squared random variables, we derive a novel, simple, and accurate analytical expression for the minimum number of samples required to achieve a desired probability of detection and false alarm. This way, the number of samples can be calculated by the energy detector with low complexity. We also propose a useful approximation for the performance of cooperative energy detection.**

***Index Terms*—Cooperative sensing, energy detector, Gaussian approximation, small sample size, spectrum sensing.**

I. Introduction

THE ENERGY detector (ED), also known as radiometer, is one of the most popular detection schemes for spectrum sensing and radar applications [1]. The false-alarm and detection performances of the ED have been widely investigated in [1]–[6], assuming that the signal to be detected is either deterministic [2] or random [1], [6], and in different channel conditions, such as additive white Gaussian noise (AWGN) [1], [2], or fading channels [3], [4]. In cognitive radio applications, the ED should determine the minimum number of samples that permits to achieve a desired ED performance [6]. Despite the results available in [1]–[6], no closed-form formulae exist for the exact calculation of the minimum number of samples required by the ED in AWGN channels. An implicit exact calculation is actually possible, but requires an iterative trial-based approach, which is computationally costly for the ED. On the other hand, approximated formulae for the minimum number of samples are available only for large sample sizes [1]–[5].

This letter proposes novel analytical closed-form expressions for the minimum number of samples required by the ED in AWGN to achieve a desired probability of detection and false alarm, when detecting Gaussian or deterministic signals. The proposed expressions are obtained by exploiting a known approximation for chi-square distributions [7]. The main features of the proposed formulae are: (a) great accuracy for every sample size; (b) low computational complexity that can be tolerated by a simple ED. Besides, we propose a useful approximation for the performance of a cooperative sensing scheme based on hard decisions taken by multiple EDs.

Manuscript received June 17, 2013. The associate editor coordinating the review of this letter and approving it for publication was T. Tsiftsis.

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Digital Object Identifier 10.1109/LCOMM.2013.080813.131399

II. Exact Performance

We consider a secondary user of a cognitive radio network that employs an ED for spectrum sensing purposes. In AWGN, the received signal $\mathbf{y} = [y_1, \ldots, y_N]^T$ is expressed by

$$\mathbf{y} = \alpha \mathbf{s} + \mathbf{w}, \tag{1}$$

$$y_i = \alpha s_i + w_i, \quad i = 1, \ldots, N, \tag{2}$$

where $\mathbf{s} = [s_1, \ldots, s_N]^T$ is the primary user signal, N is the number of samples, $\mathbf{w} = [w_1, \ldots, w_N]^T$ is the complex AWGN, with zero mean and covariance $\mathbf{\Sigma}_{\mathrm{W}} = \sigma_{\mathrm{W}}^2 \mathbf{I}_N$, and $\alpha = 0$ (H_0 hypothesis) or $\alpha = 1$ (H_1 hypothesis) in the absence or presence, respectively, of the primary signal. In this section and in Sections III and IV, we assume a statistical model for the primary signal $\mathbf{s}$, which is complex Gaussian with zero mean and covariance $\mathbf{\Sigma}_{\mathrm{S}} = \sigma_{\mathrm{S}}^2 \mathbf{I}_N$ (a deterministic model for the primary signal will be considered in Section V).

The ED first computes the test statistic

$$T(\mathbf{y}) = \|\mathbf{y}\|^2 = \sum_{i=1}^{N} |y_i|^2. \tag{3}$$

If $T(\mathbf{y})$ exceeds the threshold t, then the ED decides that the signal is present, otherwise it decides for the signal absence. Under the H_0 hypothesis, $2T(\mathbf{y})/\sigma_{\mathrm{W}}^2$ is a chi-squared random variable with $2N$ degrees of freedom, and hence the probability of false alarm is [1], [6],

$$P_{\mathrm{FA}} = \Pr\{T(\mathbf{y}) > t \,|\, \alpha = 0\} = 1 - F_{2N}(2t/\sigma_{\mathrm{W}}^2), \tag{4}$$

where $F_{2N}(x) = [\Gamma(N)]^{-1} \int_0^{x/2} \nu^{N-1} e^{-\nu}\, d\nu$ is the regularized lower incomplete gamma function, and $\Gamma(N)$ is the gamma function [7]. Under the H_1 hypothesis, $2T(\mathbf{y})/(\sigma_{\mathrm{S}}^2 + \sigma_{\mathrm{W}}^2)$ is a chi-squared random variable with $2N$ degrees of freedom, and hence the probability of detection is [1], [6],

$$P_{\mathrm{D}} = \Pr\{T(\mathbf{y}) > t \,|\, \alpha = 1\} = 1 - F_{2N}\left(\frac{2t}{\sigma_{\mathrm{S}}^2 + \sigma_{\mathrm{W}}^2}\right). \tag{5}$$

From (4) and (5), by eliminating the threshold t, the receiver operating characteristic (ROC) can be obtained as

$$P_{\mathrm{D}} = 1 - F_{2N}\left((1+\gamma)^{-1} F_{2N}^{-1}(1 - P_{\mathrm{FA}})\right), \tag{6}$$

where $\gamma = \sigma_{\mathrm{S}}^2/\sigma_{\mathrm{W}}^2$ is the signal-to-noise ratio (SNR) and $x = F_{2N}^{-1}(p)$ is the inverse of $p = F_{2N}(x)$ with respect to x.

In order to exactly calculate the minimum number of samples N that is required to achieve a prescribed performance $(P_{\mathrm{FA}}, P_{\mathrm{D}})$ for a given SNR γ, an iterative algorithm is necessary, since (6) cannot be inverted with respect to N. For instance, the iterative algorithm can calculate the right-hand side of (6) for increasing values of N, until the right-hand side becomes equal to (or larger than) the prescribed P_{D}. However, this iterative approach has a large computational complexity, since it requires multiple evaluations of two-dimensional functions such as $F_{2N}(x)$ and $F_{2N}^{-1}(p)$. If the ED has to calculate the required number of samples, it is necessary to use low-complexity (but accurate) analytical expressions that are capable to explicitly provide the required minimum number of samples N in closed form.

III. Gaussian Approximation

In order to find the required number of samples N, the standard approach is to approximate the chi-squared distribution as Gaussian [1], [5]. The central limit theorem yields [1]

$$F_{2N}(x) \approx 1 - Q\left(\frac{x-2N}{2\sqrt{N}}\right), \quad (7)$$

where $Q(x) = \int_x^\infty e^{-\nu^2/2}\, d\nu/\sqrt{2\pi}$. Using (7), (6) becomes

$$P_{\mathrm{D}} \approx Q\left((1+\gamma)^{-1}Q^{-1}(P_{\mathrm{FA}}) - \gamma(1+\gamma)^{-1}\sqrt{N}\right), \quad (8)$$

which permits to analytically derive the minimum number of samples that are required to achieve a prescribed performance $(P_{\mathrm{FA}}, P_{\mathrm{D}})$ for a given SNR γ, as expressed by

$$N \approx \left\lceil \left[\gamma^{-1}Q^{-1}(P_{\mathrm{FA}}) - (1+\gamma^{-1})Q^{-1}(P_{\mathrm{D}})\right]^2 \right\rceil. \quad (9)$$

Since it is obtained using the central limit theorem, (9) is a good estimate of the required number N of samples only when N is sufficiently high. In addition, Gaussian approximations tend to be more accurate in the middle of a bell-shaped probability density function (pdf), rather than near to the edges or queues of a pdf. Hence, some care should be taken when using (9) with very low P_{FA} or very high P_{D}.

IV. Cube-of-Gaussian Approximation

To avoid the abovementioned shortcomings of the Gaussian approximation, we make use of the cube-of-Gaussian approximation approach [1], [8]. To the best of our knowledge, this approach is novel and has not been exploited for characterizing the performance of the ED.

In the proposed cube-of-Gaussian approach, a chi-squared random variable, divided by $2N$ (which is the number of degrees of freedom), is approximated by the cube of a Gaussian random variable [8] with mean $1-(9N)^{-1}$ and variance $(9N)^{-1}$. This is summarized by

$$F_{2N}(x) \approx 1 - Q\left(\frac{\sqrt[3]{x/(2N)} - [1-(9N)^{-1}]}{(9N)^{-1/2}}\right). \quad (10)$$

Using (10), the ROC (6) becomes

$$P_{\mathrm{D}} \approx Q\left(f(\gamma)Q^{-1}(P_{\mathrm{FA}}) - [1-f(\gamma)]g(N)\right), \quad (11)$$

$$f(\gamma) = (1+\gamma)^{-1/3}, \qquad g(N) = \frac{9N-1}{3\sqrt{N}}. \quad (12)$$

The approximated ROC (11) permits to derive the minimum number of samples that are required to achieve a desired performance $(P_{\mathrm{FA}}, P_{\mathrm{D}})$ for a given SNR γ, as expressed by

$$N \approx \left\lceil \frac{1}{36}\left(b + \sqrt{b^2+4}\right)^2 \right\rceil, \quad (13)$$

$$b = b(\gamma, P_{\mathrm{FA}}, P_{\mathrm{D}}) = \frac{f(\gamma)Q^{-1}(P_{\mathrm{FA}}) - Q^{-1}(P_{\mathrm{D}})}{1-f(\gamma)}, \quad (14)$$

with $f(\gamma)$ defined in (12). Since the cube-of-Gaussian approximation (10) of a chi-squared pdf is more accurate than the classical Gaussian approximation (7) [8], we expect that (13) will be more accurate than (9) for small sample sizes and for very low P_{FA} (or very high P_{D}). This will be confirmed by the numerical results reported in Section VII.

Note that, when the SNR γ tends to zero, b in (14) tends to $3[Q^{-1}(P_{\mathrm{FA}}) - Q^{-1}(P_{\mathrm{D}})]/\gamma$, and N in (13) tends to $b^2/9$. Therefore, at very low SNR, the number N of required samples tends to $[Q^{-1}(P_{\mathrm{FA}}) - Q^{-1}(P_{\mathrm{D}})]^2/\gamma^2$, which is the same asymptotic result obtained from (9) in the Gaussian case. This means that the cube-of-Gaussian approximation (13) agrees with the Gaussian approximation (9) at very low SNR, or equivalently when the number of samples is very high.

The cube-of-Gaussian approximation (10) can be used also for identifying a specific point of the ROC when N and γ are fixed. For instance, we can choose the couple $(P_{\mathrm{FA}}, P_{\mathrm{D}})$ that minimizes the total error rate (TER), defined as

$$P_{\mathrm{TER}} = P_{\mathrm{FA}} + (1-P_{\mathrm{D}}). \quad (15)$$

From (15), the minimum TER is obtained when $\partial P_{\mathrm{D}}/\partial P_{\mathrm{FA}} = 1$, which, after some computations using (10)–(12), leads to

$$P_{\mathrm{FA}} \approx Q\big(a(\gamma, N)\big), \quad (16)$$

$$P_{\mathrm{D}} \approx Q\big(f(\gamma)a(\gamma,N) - [1-f(\gamma)]g(N)\big), \quad (17)$$

$$a(\gamma, N) = \sqrt{\left(\frac{g(N)}{1+f(\gamma)}\right)^2 + \frac{2\ln(1/f(\gamma))}{1-[f(\gamma)]^2}} - \frac{f(\gamma)g(N)}{1+f(\gamma)}. \quad (18)$$

It is noteworthy that, differently from the considered Gaussian and cube-of-Gaussian approximations, a chi-squared random variable can also be conveniently approximated as the square, or the fourth-power, of a Gaussian random variable with suitable mean and variance [8], [9]. We only consider the cubic approximation, which is the one that gives the best accuracy [8], [9]; however, a similar analysis could be performed, e.g., for square-of-Gaussian approximations.

V. Deterministic Model

In Section II, we have assumed a statistical model for the primary user signal $\mathbf{s}$, since, in cognitive radio applications, the primary signal contains information and is therefore random. Specifically, the Gaussian assumption for the pdf of $\mathbf{s}$ is compliant with a primary user that transmits multicarrier signals. However, for deterministic signals, and for single-carrier transmissions with constant envelope signals, a deterministic model for the primary user signal $\mathbf{s}$ may be more appropriate. In this section, we show that the cube-of-Gaussian approximation approach can be used also for deterministic signals.

We redefine the SNR $\mu = E_{\mathrm{S}}/(N\sigma_{\mathrm{W}}^2)$, where $E_{\mathrm{S}} = \|\mathbf{s}\|^2$. Assuming a primary signal with constant modulus $|s_i| = s$, we have $E_{\mathrm{S}} = Ns^2$ and the SNR becomes $\mu = s^2/\sigma_{\mathrm{W}}^2$. When the signal is absent, the probability of false alarm is obviously expressed by (4). Under the H_1 hypothesis, $2T(\mathbf{y})/\sigma_{\mathrm{W}}^2$ in (3) is a noncentral chi-square random variable [7], [10], with $2N$ degrees of freedom and noncentrality parameter $\lambda = 2\mu N$. This noncentral chi-square random variable can be well approximated as a chi-square random variable with an increased number of degrees of freedom [2], [10], which is expressed by $2(1+\mu)N$. As a consequence, we can again apply the cube-of-Gaussian approach of Section IV. Due to the lack of space, we omit the whole derivation, and herein report the ROC as

$$P_{\rm D} \approx Q\left(f_{2/3}(\mu)[Q^{-1}(P_{\rm FA})+g(N)] - \frac{9N[f_1(\mu)]^2-1}{3\sqrt{N}f_1(\mu)}\right),$$
$$f_x(\mu) = (1+\mu)^x/(1+2\mu)^{1/2}, \tag{19}$$

and the minimum number of samples to achieve a specified performance $(P_{\rm FA}, P_{\rm D})$ for a given SNR μ, as

$$N \approx \left\lceil \frac{1}{4A^2}\left(-B+\sqrt{B^2-4AC}\right)^2 \right\rceil, \tag{20}$$
$$A = 9[f_1(\mu) - f_{2/3}(\mu)], \tag{21}$$
$$B = 3[Q^{-1}(P_{\rm D}) - f_{2/3}(\mu)Q^{-1}(P_{\rm FA})], \tag{22}$$
$$C = f_{2/3}(\mu) - [f_1(\mu)]^{-1}. \tag{23}$$

Differently, in the deterministic case, the Gaussian approximation produces [5]

$$N \approx \left\lceil \left[\frac{1}{\mu}Q^{-1}(P_{\rm FA}) - \frac{\sqrt{1+2\mu}}{\mu}Q^{-1}(P_{\rm D})\right]^2 \right\rceil. \tag{24}$$

VI. Cooperative Energy Detection

In this section, we extend the proposed analysis to centralized spectrum sensing by means of K cooperating sensors, each one equipped with an ED [11], [12]. We assume the same scenario of [11], where the distance between the primary transmitter and any cognitive radio is large when compared to the distance between any two cognitive radios. This assumption leads to EDs with the same SNR [11]. Similarly to [11], for the primary signal we consider the deterministic model of Section V, and assume that the K EDs use the same threshold t. Under these assumptions, the K EDs have the same probabilities of detection and false alarm [11]. A fusion center (FC) collects the K hard decisions taken by the K sensors, and decides by means of either a majority-voting rule or another voting rule [11], such as the "and" rule or the "or" rule. Assuming a majority-voting rule and K odd, the probability of false alarm $\Pi_{\rm FA} = h_K(P_{\rm FA})$ at the FC, or the probability of detection $\Pi_{\rm D} = h_K(P_{\rm D})$ at the FC, is expressed as [11], [12],

$$\Pi = h_K(P) = \sum_{i=\frac{K+1}{2}}^{K} \binom{K}{i} P^i(1-P)^{(K-i)}$$
$$= I_P\left(\frac{K+1}{2}, \frac{K+1}{2}\right), \tag{25}$$

where $I_x(y,z)$ is the regularized incomplete beta function [7]. If the number of sensors K is large enough, we can approximate the binomial distribution in (25) by a Gaussian, as expressed by

$$\Pi = h_K(P) \approx Q\left(\frac{\sqrt{K}(1-2P)}{2\sqrt{P(1-P)}}\right), \tag{26}$$
$$P = h_K^{-1}(\Pi) \approx \frac{1}{2} - \frac{Q^{-1}(\Pi)}{2\sqrt{K+[Q^{-1}(\Pi)]^2}}. \tag{27}$$

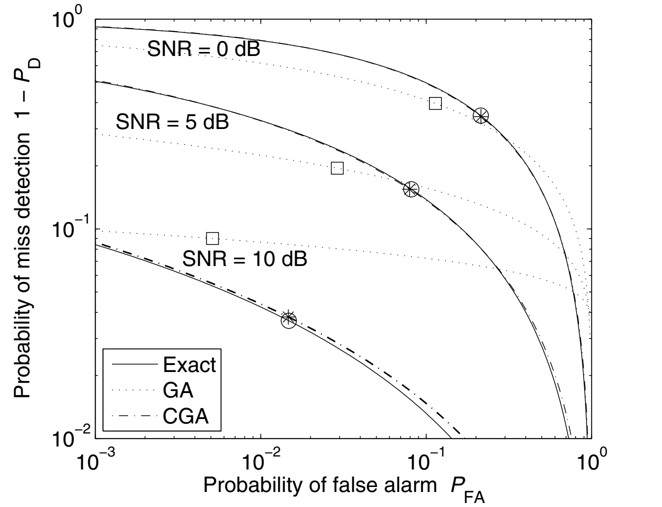

Fig. 1. Complementary ROC of the ED for the statistical model ($N = 3$). The markers denote the minimum TER points.

To achieve a prescribed FC performance $(\Pi_{\rm FA}, \Pi_{\rm D})$ for a given number of sensors K, we can determine the required sensor performance $P_{\rm FA} = h_K^{-1}(\Pi_{\rm FA})$ and $P_{\rm D} = h_K^{-1}(\Pi_{\rm D})$ using the Gaussian approach (27), and then the number of samples N required by each ED by the cube-of-Gaussian approach (20). Alternatively to (27), we can determine the required sensor performance $P_{\rm FA}$ and $P_{\rm D}$ by exploiting the improved approximation 26.5.22 in [7] for the inverse of the regularized incomplete beta function in (25), as expressed by

$$P = h_K^{-1}(\Pi) \approx \frac{1}{1+e^{2K^{-1}Q^{-1}(\Pi)\sqrt{K+\frac{[Q^{-1}(\Pi)]^2-3}{6}}}}, \tag{28}$$

and then use again (20) to identify the number of samples N required by each ED.

On the other hand, if both the FC performance $(\Pi_{\rm FA}, \Pi_{\rm D})$ and the sensor performance $(P_{\rm FA}, P_{\rm D})$ are fixed, we can determine the required number of sensors K as

$$K = \max\{\bar{K}(\Pi_{\rm FA}, P_{\rm FA}), \bar{K}(\Pi_{\rm D}, P_{\rm D})\}, \tag{29}$$

where, by the Gaussian approximation (26)–(27),

$$\bar{K}(\Pi, P) \approx \left\lceil 4P(1-P)\left[\frac{Q^{-1}(\Pi)}{1-2P}\right]^2 \right\rceil, \tag{30}$$

or, alternatively, by the improved approximation (28),

$$\bar{K}(\Pi, P) \approx \left\lceil \frac{\bar{B}(\Pi) + \sqrt{\left[\bar{B}(\Pi)\right]^2 + 4\bar{A}(P)\bar{C}(\Pi)}}{2\bar{A}(P)} \right\rceil, \tag{31}$$
$$\bar{A}(P) = \left[\ln\sqrt{(1-P)/P}\right]^2, \tag{32}$$
$$\bar{B}(\Pi) = [Q^{-1}(\Pi)]^2, \tag{33}$$
$$\bar{C}(\Pi) = [Q^{-1}(\Pi)]^2\left\{[Q^{-1}(\Pi)]^2 - 3\right\}/6. \tag{34}$$

VII. Numerical Results

We verify the accuracy of the proposed approximations by means of numerical examples. Fig. 1 shows the complementary ROC of the ED, for the statistical model of Section II,

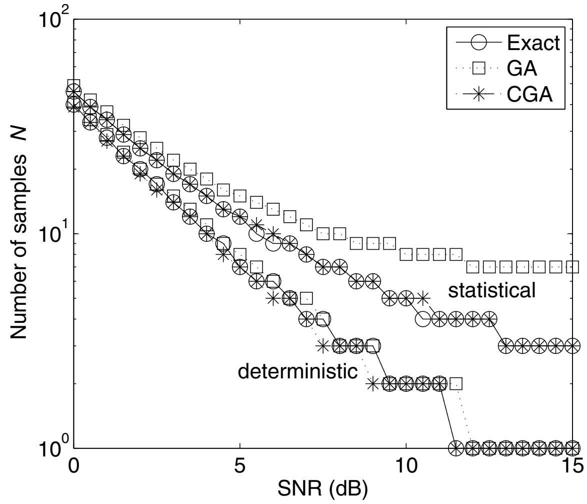

Fig. 2. Number of samples N required for $P_{\mathrm{FA}} = 1 - P_{\mathrm{D}} = 10^{-2}$.

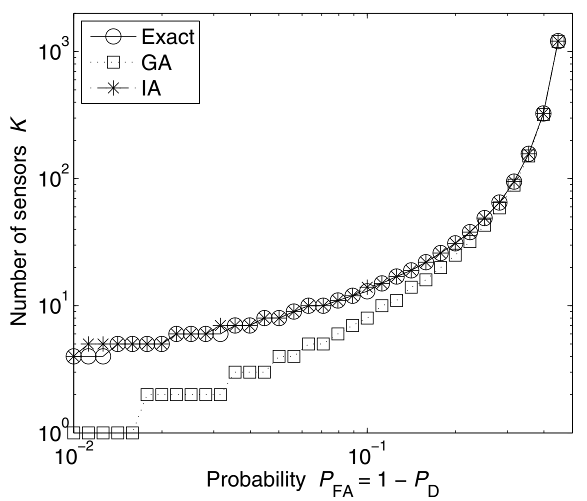

Fig. 3. Number of sensors K required for $\Pi_{\mathrm{FA}} = 1 - \Pi_{\mathrm{D}} = 10^{-4}$.

when the number of samples is $N = 3$. In this case, the proposed cube-of-Gaussian approximation (11), denoted with CGA, agrees with the exact result (6), especially at low SNR. On the contrary, the classical Gaussian approximation (8), denoted with GA, significantly deviates from the exact result (6). Besides, Fig. 1 clearly highlights that the proposed cube-of-Gaussian approach correctly locates the minimum-TER point.

Fig. 2 illustrates the number of samples N required by the ED to achieve the performance $P_{\mathrm{FA}} = 1 - P_{\mathrm{D}} = 10^{-2}$, as a function of the SNR. Again, the proposed cube-of-Gaussian approximation well matches the exact result, which has been obtained iteratively. For the statistical model of Section II, the classical Gaussian approximation is accurate only when the number of samples is high, i.e., only at low SNR. For deterministic signals, the Gaussian approximation is noticeably more accurate than for the statistical model.

We now consider a cooperative scenario. Fig. 3 displays the number of sensors K that are required by the FC to achieve the performance $\Pi_{\mathrm{FA}} = 1 - \Pi_{\mathrm{D}} = 10^{-4}$, as a function of the performance $P_{\mathrm{FA}} = 1 - P_{\mathrm{D}}$ of each local ED. Clearly, the Gaussian approximation (30), denoted with GA, becomes accurate only when there are many sensors, whereas the proposed improved approximation (31)–(34), denoted with IA, is very accurate also with very few sensors.

VIII. Conclusions

We have proposed, discussed, and verified, simple and accurate approximations to be used for the performance evaluation of both the ED and a cooperative ED in AWGN channels. We believe that the proposed approximations could be exploited to compute the minimum number of samples requested by the ED in fading channels [3]–[5].

Acknowledgment

This research has been funded by the Fondazione Cassa di Risparmio di Perugia, within the project ”Algoritmi cooperativi per reti wireless di sensori,” with code 2010.011.0417.

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