Ionospheric calibration from an array signal processing perspective

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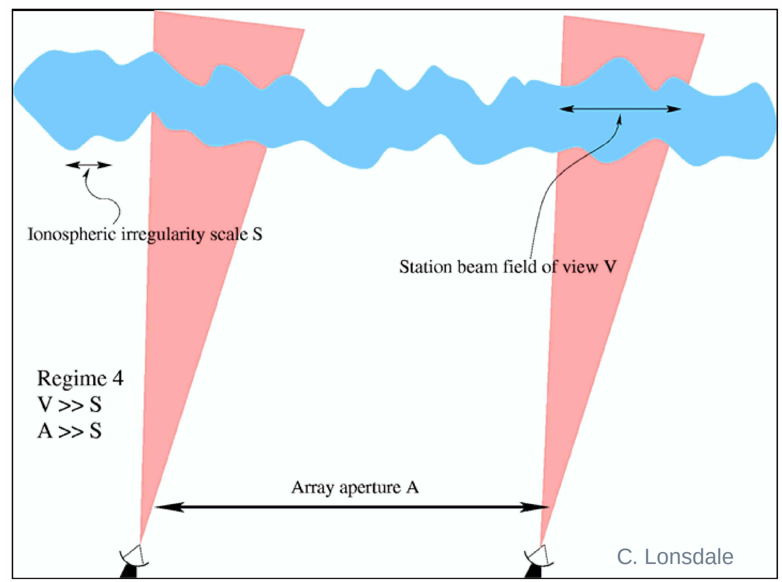
Ionospheric Calibration

The most difficult ionospheric calibration problem arises when both the field of view and the array aperture are larger than the ionospheric irregularity scale. The complex gains depend on

- Station
- Look direction
- Time



Current Calibration Schemes

The current approach can be summarized as:

1. Create a deterministic data model of the form

$$\mathbf{x} = \mathbf{f}(\boldsymbol{\theta}) + \mathbf{w},$$

   where $\mathbf{x}$ are the observation, $\mathbf{f}()$ is the Measurement Equation, $\boldsymbol{\theta}$ are the unknowns, and $\mathbf{w}$ is the noise.

2. Use a Least Squares solver to find

$$\hat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \|\mathbf{x} - \mathbf{f}(\boldsymbol{\theta})\|^2,$$

   where $\hat{\boldsymbol{\theta}}$ is an estimate of $\boldsymbol{\theta}$.

Rationale behind Least Squares I

Estimation theory tells us that best estimator for a deterministic data model is the Minimum Variance Unbiased (MVU) estimator. Its variance is given by the Cramer-Rao bound.

Unfortunately the MVU can be hard to find or does not even exist. An estimator that is asymptotically efficient and always exists is Maximum Likelihood

$$\hat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}} p(\mathbf{x}, \boldsymbol{\theta})$$

Rationale behind Least Squares II

In the case of Gaussian noise, Maximum Likelihood simplifies to a (Weighted) Least Squares fit

$$
\begin{aligned}
\hat{\theta} &= \arg\max_{\boldsymbol{\theta}} p(\mathbf{x}, \boldsymbol{\theta}) = \arg\max_{\boldsymbol{\theta}} \frac{1}{(2\pi)^{\frac{k}{2}} |\mathbf{C}_{\mathbf{w}}|^{\frac{1}{2}}} \exp\left[-\frac{1}{2} (\mathbf{x} - \mathbf{f}(\boldsymbol{\theta}))^{\mathrm{T}} \mathbf{C}_{\mathbf{w}}^{-1} (\mathbf{x} - \mathbf{f}(\boldsymbol{\theta})) \right] \Rightarrow \\
\hat{\theta} &= \arg\min_{\boldsymbol{\theta}} \| \mathbf{C}_{\mathbf{w}}^{-\frac{1}{2}} (\mathbf{x} - \mathbf{f}(\boldsymbol{\theta})) \|^2
\end{aligned}
$$

In radio astronomy we usually deal with a large number of samples and Gaussian noise. In that case Least Squares is an efficient estimator.

Ionospheric Modeling

In absence of (traveling) disturbances, the dynamics of the ionosphere can be described by turbulent flow. A deterministic model of turbulence will be too complex for calibration purposes.

A far simpler, yet effective model of turbulence is that of a random process.

- Stochastic model for ionosphere
- Estimators for solving stochastic models

Structure Function

A random process is specified by first and second order moments.

- First order moment is the mean
- Second order moments can be specified as covariance, *structure function* or power spectral density

The structure function is defined as

$$D_x(|\mathbf{x}_1 - \mathbf{x}_2|) = \mathrm{E}[(x(\mathbf{r}_1) - x(\mathbf{x}_2))^2]$$

Kolmogorov Turbulence

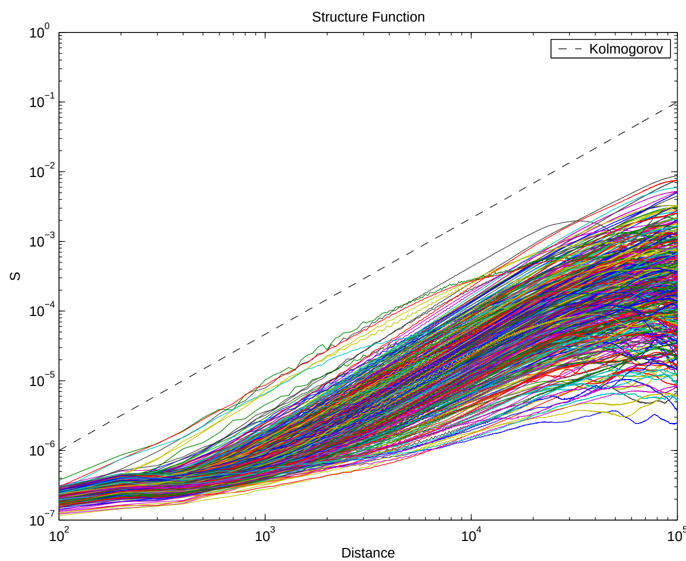
By means of a dimensional analysis, Kolmogorov has derived the 3-D power spectral density of turbulent flow. It obeys a powerlaw with exponent $\beta = -11/3$.

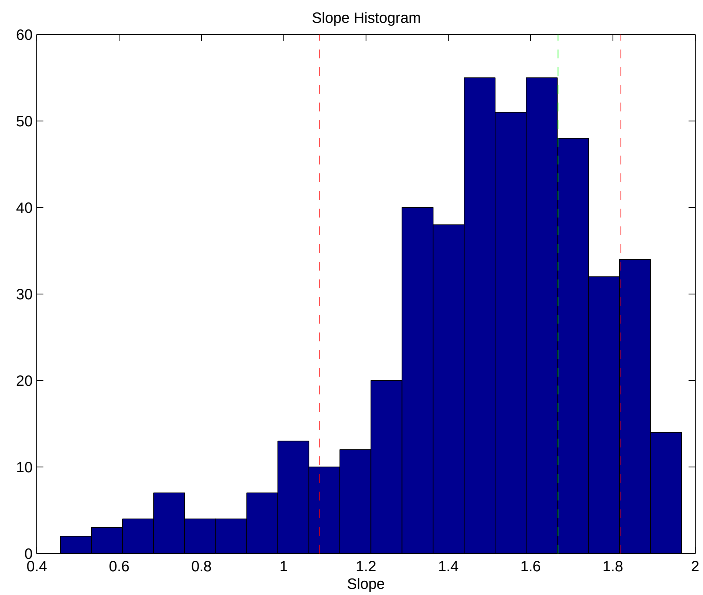
The fluctuations of ionospheric electron density follow the same spectral density. Integrating out the vertical dimension leads to the Singel Layer Model. The structure function of the Total Electron Content is

$$D_{\mathrm{T}EC}(r) \sim r^{5/3}$$

Results GPS measurement

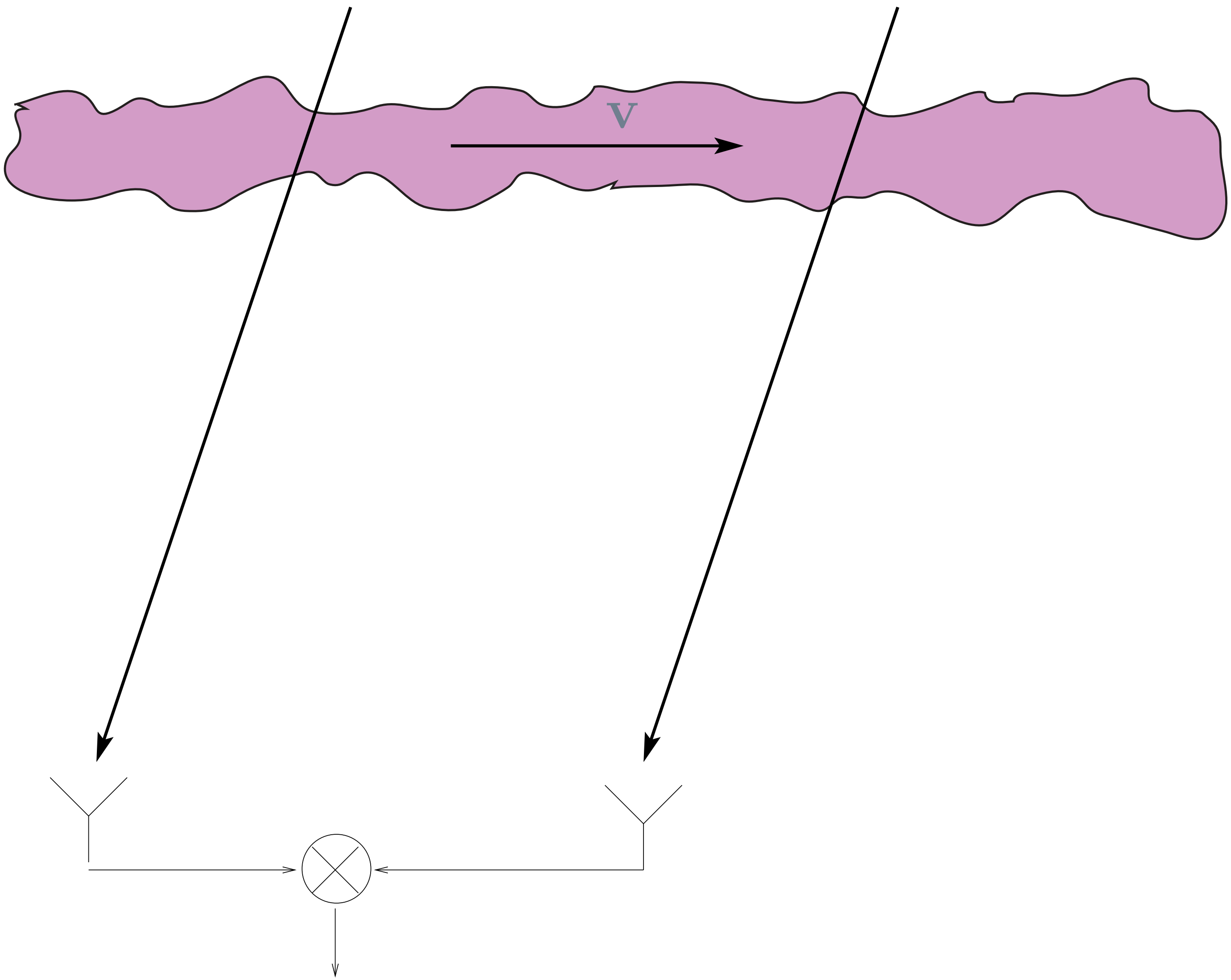
The following estimates of the structure function are based on GPS data measured in The Netherlands, last January.





The green dashed line is the location of the predicted 5/3

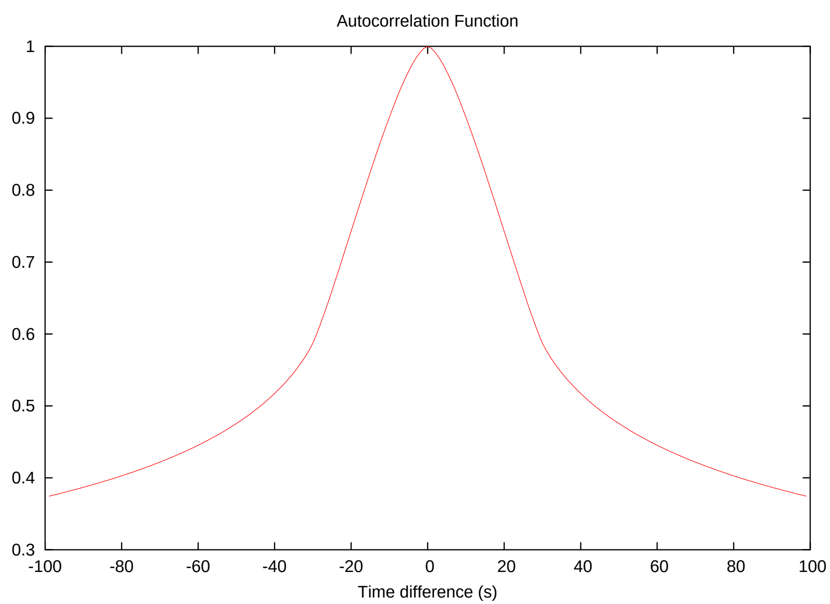
Frozen flow



Frozen flow –> Correlations

From the structure function in combination with the frozen flow model we can derive the autocorrelation function of the phase difference

$$\kappa(\tau) = \frac{1}{2}D(\|\mathbf{b} + \mathbf{v}\tau\|) + \frac{1}{2}D(\|\mathbf{b} - \mathbf{v}\tau\|) - D(\mathbf{v}\tau)$$



Estimation based on stochastic mode

The correlation function is a quantitative measure of smoothness
We can use it to find the optimal basis. The eigenvalue decompos
tion of $\mathbf{C}_{\boldsymbol{\theta}}$ is given by

$$\mathbf{C}_{\boldsymbol{\theta}} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{\mathrm{T}} = \begin{bmatrix} | & & | \\ \mathbf{u}_1 & \dots & \mathbf{u}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} -\mathbf{u}_1^{\mathrm{T}}- \\ \vdots \\ -\mathbf{u}_n^{\mathrm{T}}- \end{bmatrix}$$

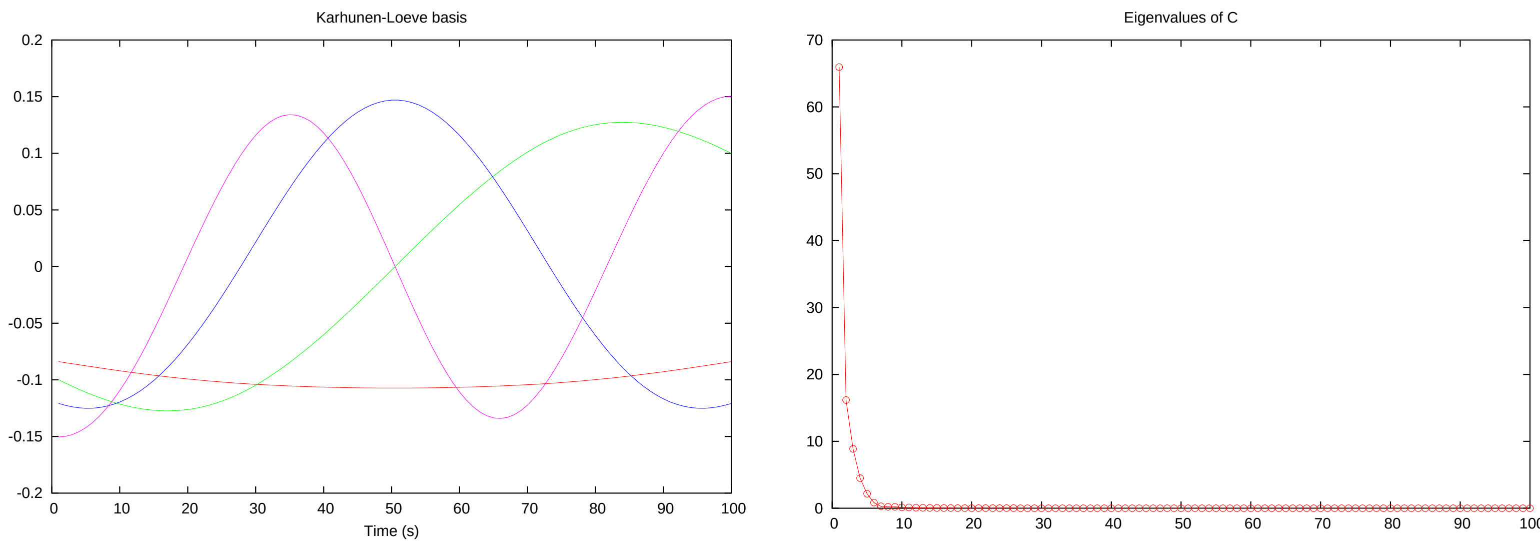
The Karhunen-Loève transformation is given by

$$\boldsymbol{\theta} = \mathbf{U}\mathbf{p}, \quad \mathbf{p} = \mathbf{U}^{\mathrm{T}}\boldsymbol{\theta}$$

The correlation matrix of the tranformed data is

$$\mathbf{C}_{\mathbf{p}} = \boldsymbol{\Lambda}$$

Karhunen-Loève basis



Now we can apply LS as before, but even using the KL basis the result is not optimal.

MMSE estimator

The estimator with the lowest Mean Square Error is the Bayesian MSSE

$$\hat{\boldsymbol{\theta}} = \arg\min_{\hat{\boldsymbol{\theta}}} \mathrm{E}[|\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}|^2]$$

The solution of this minimization problem is given by

$$\hat{\boldsymbol{\theta}} = \mathrm{E}[\boldsymbol{\theta}|\mathbf{x}]$$

This is the expected value of $\boldsymbol{\theta}$ with respect to the A Posteriori PDF.

A Posteriori PDF

The A Posteriori PDF is found using Bayes rule

$$p(\boldsymbol{\theta}|\mathbf{x}) = \frac{p(\mathbf{x}|\boldsymbol{\theta})p(\boldsymbol{\theta})}{\int p(\mathbf{x}|\boldsymbol{\theta})p(\boldsymbol{\theta})d\boldsymbol{\theta}}$$

Evaluation of the expected value is usually too complex. Instead we can search for the maximum of the a posteriori pdf.

$$\hat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta}} p(\boldsymbol{\theta}|\mathbf{x})$$

Maximum A Posteriori Estimator

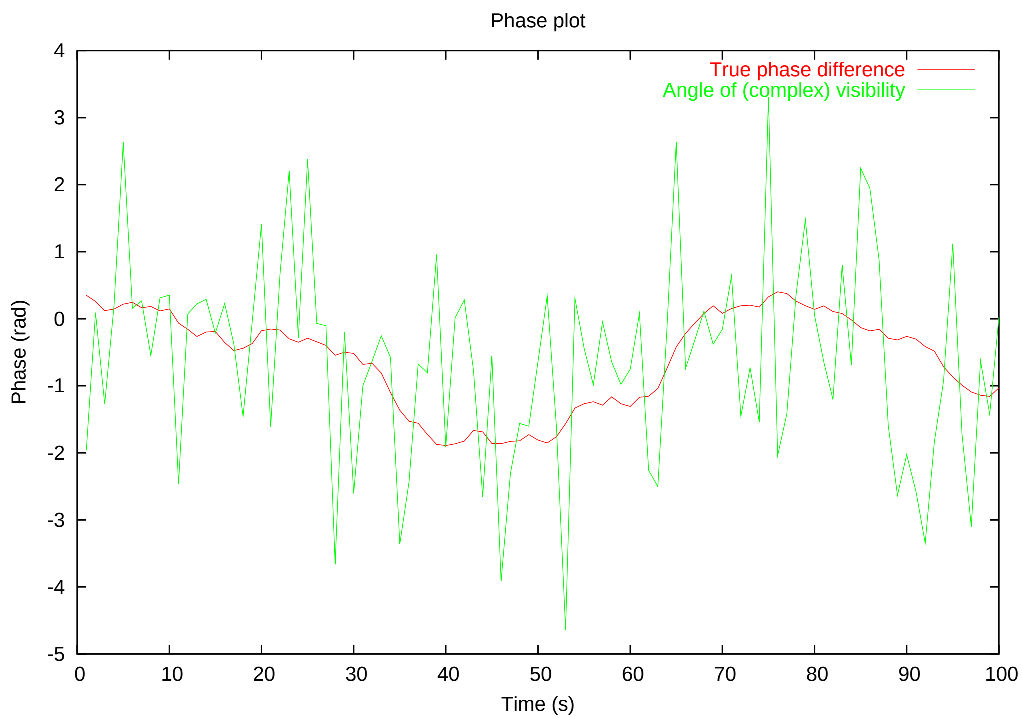
The MAP is very similar to ML

$$
\begin{aligned}
\hat{\boldsymbol{\theta}} &= \arg\max_{\boldsymbol{\theta}} p(\mathbf{x}, \boldsymbol{\theta}) \\
&= \arg\max_{\boldsymbol{\theta}} \frac{1}{(2\pi)^{\frac{k}{2}} |\mathbf{C}_{\mathbf{w}}|^{\frac{1}{2}}} \exp\left[-\frac{1}{2}(\mathbf{x} - \mathbf{f}(\boldsymbol{\theta}))^{\mathrm{T}} \mathbf{C}_{\mathbf{w}}^{-1} (\mathbf{x} - \mathbf{f}(\boldsymbol{\theta}))\right] \\
&\quad \frac{1}{(2\pi)^{\frac{n}{2}} |\mathbf{C}_{\boldsymbol{\theta}}|^{\frac{1}{2}}} \exp\left[-\frac{1}{2}(\boldsymbol{\theta}^{\mathrm{T}} \mathbf{C}_{\boldsymbol{\theta}}^{-1} \boldsymbol{\theta})\right] \Rightarrow \\
\hat{\boldsymbol{\theta}} &= \arg\min_{\boldsymbol{\theta}} \|\mathbf{C}_{\mathbf{w}}^{-\frac{1}{2}} (\mathbf{x} - \mathbf{f}(\boldsymbol{\theta}))\|^2 + \|\mathbf{C}_{\boldsymbol{\theta}}^{-\frac{1}{2}} \boldsymbol{\theta}\|^2
\end{aligned}
$$

Using the Karhunen Loève transform this becomes

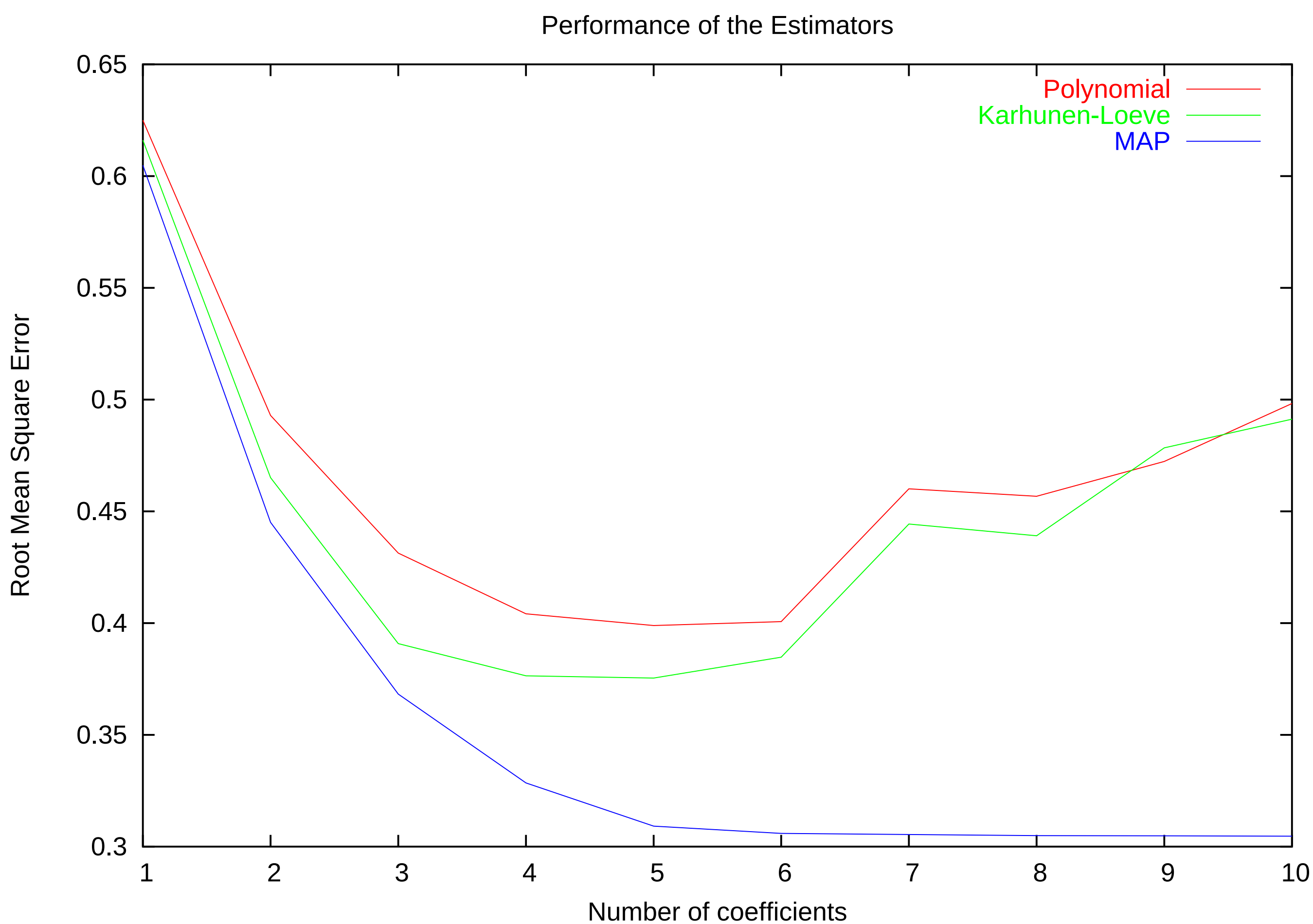
$$
\begin{aligned}
\hat{\mathbf{p}} &= \arg\min_{\mathbf{p}} \|\mathbf{C}_{\mathbf{w}}^{-\frac{1}{2}} (\mathbf{x} - \mathbf{f}(\mathbf{U}\mathbf{p}))\|^2 + \|\boldsymbol{\Lambda}^{-\frac{1}{2}} \mathbf{p}\|^2 \\
\hat{\boldsymbol{\theta}} &= \mathbf{U}\hat{\mathbf{p}}
\end{aligned}
$$

Simulations I

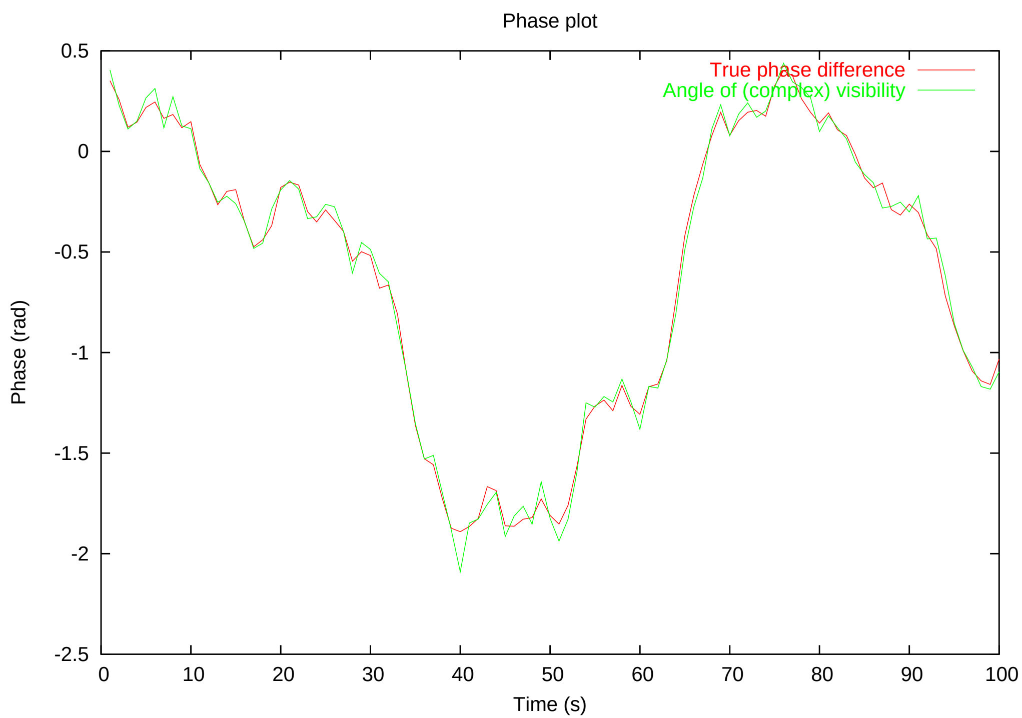


$$
\begin{aligned}
\mathrm{SNR} &= -18\mathrm{dB} \\
N &= 1000 \\
v &= 100m/s \\
b &= 3000m \\
D(3000) &= 1
\end{aligned}
$$

Simulations II

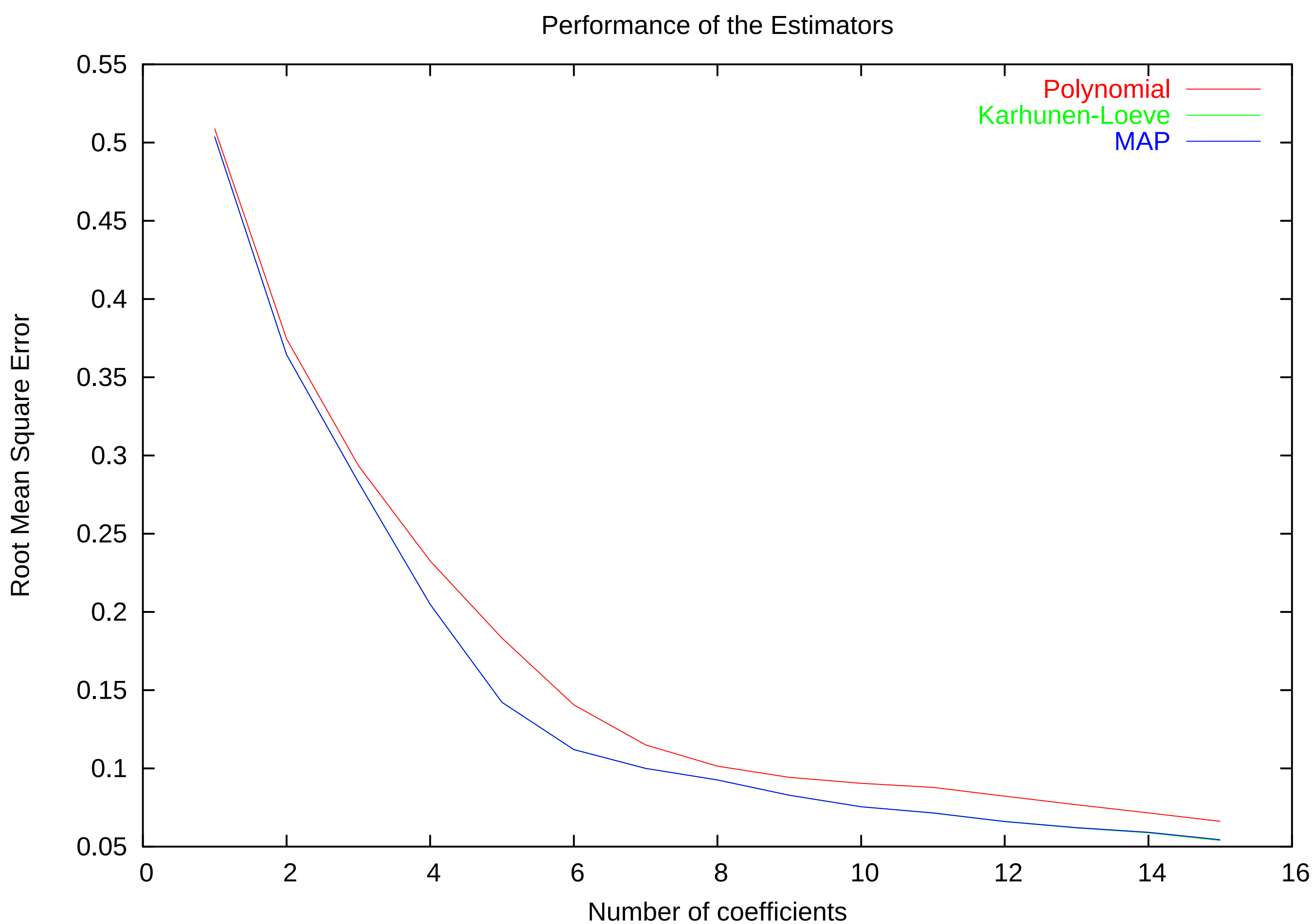


Simulations III



$$\begin{aligned} \mathrm{SNR} &= 0\mathrm{dB} \\ N &= 1000 \\ v &= 100m/s \\ b &= 3000m \\ D(3000) &= 1 \end{aligned}$$

Simulations IV



Conclusions

Advantages of stochastic modeling and MAP estimation are

- Performance close to theoretic optimum (MMSE).
- No major changes to the software needed
- Minor impact on computational complexity
- Increasing model order always reduces error

Future work

- Complete the stochastic model of the ionosphere to include space, time and frequency correlations
- Improve stochastic model using LOFAR measurements
- Extension to TID?