Near-Field Source Localization Using Sparse Recovery Techniques

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***Abstract*—Near-field source localization is an important aspect in many diverse areas such as acoustics, seismology, to list a few. The planar wave assumption frequently used in far-field source localization is no longer valid when the sources are in the near field. Near-field sources can be localized by solving a joint direction-of-arrival and range estimation problem. The original near-field source localization problem is a multi-dimensional non-linear optimization problem which is computationally intractable. In this paper, we use a grid-based model and by further leveraging the sparsity, we can solve the aforementioned problem efficiently using any of the off-the-shelf ℓ_1-norm optimization solvers. When multiple snapshots are available, we can also exploit the cross-correlations among the symmetric sensors of the array and further reduce the complexity by solving two sparse reconstruction problems of lower dimensions instead of a single sparse reconstruction problem of a higher dimension.**

I. Introduction

Source localization is an important aspect for target tracking and location-aware services, and has many applications in the field of seismology, acoustics, radar, sonar, and oceanography. Bearing or direction-of-arrival (DOA) estimation for narrowband signals is an extensively studied topic [1], [2]. DOA estimation can be categorized into two types, based on the distance between the source and the antenna array: (a) far-field (e.g., $r \gg 2D^2/\lambda$), and (b) near-field source localization, where r is the range between the source and the phase-reference of the array, D is the array aperture, and λ is the wavelength of the source signal. In far-field source localization, the wavefront of the signal impinging on the array is assumed to be planar [1], [3]. However, the curvature of the wavefront is no longer negligible when sources are located close to the array (i.e., in the near field or Fresnel region). Therefore, the algorithms that leverage the planar-wave assumptions for DOA estimation are no longer valid. In this work, we focus on near-field source localization, which is traditionally done by a joint DOA and range (distance between the source and the phase-reference of the array) estimation.

Traditional approaches to the near-field localization problem extend techniques like multiple signal classification (MUSIC) to a two-dimensional field [4]. However, the performance of the MUSIC algorithm deteriorates at low SNRs and when the sources are correlated. In [5], the wavefront is assumed to be piece-wise linear, and the uniform linear array (ULA) is divided into several subarrays. The wavefront of the signal impinging on each subarray is then assumed to be planar. By using the method proposed in [1], [3] at each subarray, the location can be estimated after gathering the DOA of each subarray. In [6]–[8], instead of using the piece-wise linear approximation, a quadratic approximation (the so-called Fresnel approximation) of the wavefront is made, which makes the wavefront neither planar nor spherical. The phase delay is no longer linear with the position of the antenna element, instead, it varies quadratically with the array position and it is characterized by the azimuth (DOA) and range of the sources (see [6] for more details). However, the array has to satisfy the Nyquist sampling rate criterion in space, i.e., the spacing between two adjacent antennas needs to be less than half a wavelength.

In this paper, we localize multiple narrowband near-field sources by jointly estimating their DOA and range. Using the sparse representation framework, we form an overcomplete basis constructed using a sampling grid that is related to the possible source locations. By doing so, the original non-linear parameter estimation problem is transformed into a linear ill-posed problem. Assuming the spatial spectrum is sparse, we can localize the sources with high resolution by solving the well-known ℓ_1-regularized least-squares optimization problem. When multiple snapshots are available, using the Fresnel approximation and assuming that the sources are uncorrelated, we can decouple the DOA and range in the correlation domain. This allows us to significantly reduce the complexity, by solving two inverse problems of smaller dimensions one by one, instead of solving one inverse problem of a higher dimension. The key contribution of this paper is this complexity reduction along with high resolution near-field source localization.

II. Problem formulation

Consider K narrowband sources present in the near field impinging on an array of $M = 2p+1$ sensors as illustrated in Fig. 1. Without loss of generality, it is assumed that the phase reference of the array is at the origin, and the sensors are placed at location indices in the range $[-p, p]$. Denoting the spacing between two adjacent sensors as δ, the position of the m-th sensor will be $m\delta$ where $m \in [-p, p]$. The signal

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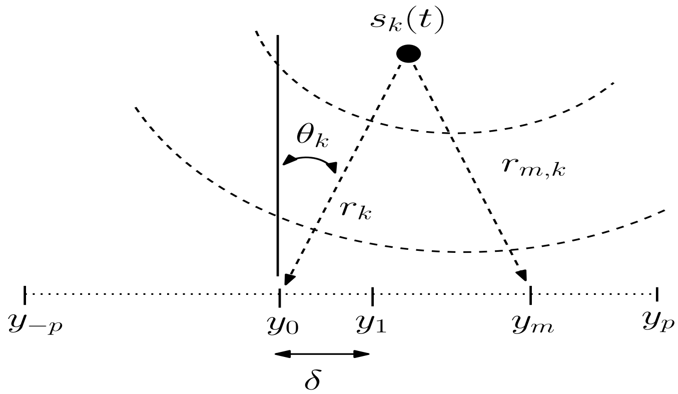

Fig. 1: A linear array receiving a signal from a near-field point source.

received by the m-th sensor at time t can be expressed as

$$y_m(t) = \sum_{k=1}^{K} s_k(t) \exp(j\frac{2\pi}{\lambda}(r_{m,k} - r_k)) + w_m(t), \quad (1)$$

where

$$r_{m,k} = \sqrt{r_k + m^2\delta^2 - 2m\delta r_k \sin(\theta_k)} \quad (2)$$

represents the distance between the m-th sensor and the k-th source, r_k is the range from the k-th source to the phase reference, $s_k(t)$ is the signal radiated by the k-th source characterized by the DOA-range pair (θ_k, r_k), λ denotes the wavelength, and $w_m(t)$ denotes the additive noise.

Stacking the measurements in $\mathbf{y}(t) = [y_{-p}(t), \ldots, y_p(t)]^T \in \mathbb{C}^{M\times 1}$, we get

$$\mathbf{y}(t) = \sum_{k=1}^{K} \mathbf{a}(\theta_k, r_k) s_k(t) + \mathbf{w}(t), \quad \text{for} \quad t = t_1, t_2, \ldots, t_T, \quad (3)$$

where T denotes the number of snapshots, $\mathbf{a}(\theta_k, r_k) \in \mathbb{C}^{M\times 1}$ is the so-called *steering vector*, and $\mathbf{w}(t) = [w_{-p}(t), \ldots, w_p(t)]^T \in \mathbb{C}^{M\times 1}$ is the noise vector. The non-linear measurement model in (3) can be concisely written as

$$\mathbf{y}(t) = \mathbf{A}(\boldsymbol{\theta}, \mathbf{r})\mathbf{s}(t) + \mathbf{w}(t), \quad (4)$$

where $\mathbf{A}(\boldsymbol{\theta}, \mathbf{r}) = [\mathbf{a}(\theta_1, r_1), \mathbf{a}(\theta_2, r_2), \ldots, \mathbf{a}(\theta_K, r_K)] \in \mathbb{C}^{M\times K}$ is the array manifold matrix, and $\mathbf{s}(t) = [s_1(t), \ldots, s_K(t)]^T \in \mathbb{C}^{K\times 1}$ is the source vector.

Problem statement (Near-field localization)**.** *Given the measurements $\mathbf{y}(t)$, and the mapping $(\boldsymbol{\theta}, \mathbf{r}) \to \mathbf{A}(\boldsymbol{\theta}, \mathbf{r})$, find the unknown locations of the near-field sources characterized by (θ_k, r_k) for all k, as well as their number K.*

III. Grid-based model

We provide a framework for localizing multiple near-field sources based on sparse reconstruction techniques. More specifically, we aim to jointly estimate the DOA $\boldsymbol{\theta} = [\theta_1, \theta_2, \ldots, \theta_K]^T$ and the range $\mathbf{r} = [r_1, r_2, \ldots, r_K]^T$. In this section, a single snapshot case is considered, with $T = 1$ in (4). The problem in (4) as it appears is a non-linear parameter estimation problem, where the matrix $\mathbf{A}(\boldsymbol{\theta}, \mathbf{r})$ depends on the unknown source locations $(\boldsymbol{\theta}, \mathbf{r})$. In order to jointly estimate both the DOA and range using the data model in (4), a multi-dimensional non-linear optimization over both $\boldsymbol{\theta}$ and $\mathbf{r}$ is required. This optimization problem is clearly computationally intractable.

Suppose all possible source locations reside in the domain $\theta_k \in [\theta_{\min}, \theta_{\max}]$ and $r_k \in [r_{\min}, r_{\max}]$ for all $k = 1, \ldots, K$. We can then cast the joint DOA-range estimation problem as a sparse reconstruction problem, where we discretize the θ-interval into N_θ and r-interval into N_r bins of resolution $\Delta\theta$ and Δr, respectively. This discretization results in an overcomplete representation of $\mathbf{A}$ in terms of the sampling grid $(\bar{\boldsymbol{\theta}}, \bar{\mathbf{r}})$ that includes all the source locations of interest with $\bar{\boldsymbol{\theta}} = [\bar{\theta}_1, \bar{\theta}_2, \ldots, \bar{\theta}_{N_\theta}]^T$ and $\bar{\mathbf{r}} = [\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_{N_r}]^T$. We construct a matrix with steering vectors corresponding to each potential source location as its columns:

$$\mathbf{A}(\bar{\boldsymbol{\theta}}, \bar{\mathbf{r}}) = [\mathbf{a}(\bar{\theta}_1, \bar{r}_1), \mathbf{a}(\bar{\theta}_1, \bar{r}_2), \ldots, \mathbf{a}(\bar{\theta}_N, \bar{r}_N)] \in \mathbb{C}^{M\times N},$$

where $N = N_\theta N_r$. The matrix $\mathbf{A}$ is now known, and does not depend on the unknown variables $(\boldsymbol{\theta}, \mathbf{r})$. Note that the number of potential source locations N will typically be much greater than the number of sources K or even the number of sensors M.

The signal is now represented by an $N \times 1$ vector $\mathbf{x}(t)$, where every source can be found as a non-zero weight $x_n(t) = s_k(t)$ if source k comes from $(\bar{\theta}_n, \bar{r}_n)$ for some k and is zero otherwise, i.e., the dominant peaks in $\mathbf{x}(t)$ correspond to the true source locations. The discrete grid-based model for a single snapshot is given by

$$\begin{aligned} \mathbf{y} &= \sum_{n=1}^{N} \mathbf{a}(\bar{\theta}_n, \bar{r}_n) x_n + \mathbf{w} \\ &= \mathbf{A}(\bar{\boldsymbol{\theta}}_n, \bar{\mathbf{r}}_n)\mathbf{x} + \mathbf{w}. \end{aligned} \quad (5)$$

This model allows us to transform the non-linear parameter estimation problem into a sparse recovery problem based on the central assumption that the vector $\mathbf{x}$ is sparse. An ideal measure for the sparsity of $\mathbf{x}$ is its ℓ_0(-quasi) norm $\|\mathbf{x}\|_0$, and mathematically we must solve for $\arg\min \|\mathbf{x}\|_0$ subject to $\|\mathbf{y} - \mathbf{A}(\bar{\boldsymbol{\theta}}_n, \bar{\mathbf{r}}_n)\|_2^2 \le \varepsilon$, where the parameter ε controls how much noise we wish to allow. However, this is a mathematically intractable combinatorial problem even for modestly sized problems. Hence, to simplify this problem we use an ℓ_1-norm regularization, which is the traditional best convex surrogate of the ℓ_0(-quasi) norm. The inverse problem can be solved using an ℓ_1-regularized least-squares (LS) methodology which is given by

$$\hat{\mathbf{x}} = \underset{\mathbf{x}\in\mathbb{C}^{N\times 1}}{\arg\min} \quad \|\mathbf{y} - \mathbf{A}(\bar{\boldsymbol{\theta}}, \bar{\mathbf{r}})\mathbf{x}\|_2^2 + \mu\|\mathbf{x}\|_1, \quad (6)$$

where μ is the sparsity regulating parameter. This optimization problem can be solved using any of the popular solvers available off-the-shelf (e.g., iterative thresholding, matching pursuit).

IV. Fresnel approximation

Using the Taylor series expansion of (2), and approximating this up to the second order, we get the so-called *Fresnel* approximation, which is given by

$$r_{m,k} \approx r_k - m\delta \sin\theta_k + m^2\delta^2 \frac{\cos^2\theta_k}{2r_k}.$$

We can now approximate $\tau_{m,k} = \frac{2\pi}{\lambda}(r_{m,k} - r_k)$ as

$$\begin{aligned} \tau_{m,k} &\approx -m\frac{2\pi\delta}{\lambda}\sin(\theta_k) + m^2\frac{\pi\delta^2}{\lambda r_k}\cos^2(\theta_k) \\ &= m\omega_k + m^2\phi_k \end{aligned} \quad (7)$$

where we re-parameterize the DOA and range, respectively as

$$\omega_k = -\frac{2\pi\delta}{\lambda}\sin(\theta_k) \quad \text{and} \quad \phi_k = \frac{\pi\delta^2}{\lambda r_k}\cos^2(\theta_k). \quad (8)$$

Using the approximation for $\tau_{m,k}$ in (1), we get

$$y_m(t) \approx \sum_{k=1}^{K} s_k(t) e^{j(m\omega_k + m^2\phi_k)} + w_m(t). \quad (9)$$

Stacking the measurements from all the M sensors, we get

$$\mathbf{y}(t) = \sum_{k=1}^{K} \tilde{\mathbf{a}}(\omega_k, \phi_k) s_k(t) + \mathbf{w}(t), \quad \text{for} \quad t = t_1, t_2, \ldots, t_T, \quad (10)$$

where $\tilde{\mathbf{a}}(\omega_k, \phi_k) = [e^{-jp\omega_k} e^{jp^2\phi_k}, \ldots, 1, \ldots, e^{jp\omega_k} e^{jp^2\phi_k}]^T \in \mathbb{C}^{M\times 1}$ is the *modified steering vector*. The output of the ULA can now be written as the following non-linear measurement model

$$\mathbf{y}(t) = \tilde{\mathbf{A}}(\boldsymbol{\omega}, \boldsymbol{\phi})\mathbf{s}(t) + \mathbf{w}(t), \quad (11)$$

where

$$\tilde{\mathbf{A}}(\boldsymbol{\omega}, \boldsymbol{\phi}) = [\tilde{\mathbf{a}}(\omega_1, \phi_1), \tilde{\mathbf{a}}(\omega_2, \phi_2), \ldots, \tilde{\mathbf{a}}(\omega_K, \phi_K)] \in \mathbb{C}^{M\times K}$$

is the array manifold, and $\mathbf{s}(t)$ and $\mathbf{w}(t)$ are the source and noise vectors, respectively.

As before, we can construct an overcomplete representation also for $\tilde{\mathbf{A}}$ using the sampling grid $(\bar{\boldsymbol{\theta}}, \bar{\mathbf{r}})$ that includes all possible source locations. This discretization results in a known matrix with steering vectors corresponding to each potential source location as its columns:

$$\tilde{\mathbf{A}}(\bar{\boldsymbol{\omega}}, \bar{\boldsymbol{\phi}}) = [\tilde{\mathbf{a}}(\bar{\omega}_1, \bar{\phi}_1), \tilde{\mathbf{a}}(\bar{\omega}_1, \bar{\phi}_2), \ldots, \tilde{\mathbf{a}}(\bar{\omega}_N, \bar{\phi}_N)] \in \mathbb{C}^{M\times N},$$

where $\bar{\omega}_n = -\frac{2\pi\delta}{\lambda}\sin(\bar{\theta}_n)$ and $\bar{\phi}_n = \frac{\pi\delta^2}{\lambda \bar{r}_n}\cos^2(\bar{\theta}_n)$ for all $n \in \{1, \ldots, N\}$. The discrete grid-based model is finally given by

$$\mathbf{y}(t) = \tilde{\mathbf{A}}(\bar{\boldsymbol{\omega}}, \bar{\boldsymbol{\phi}})\mathbf{x}(t) + \mathbf{w}(t), \quad (12)$$

and the corresponding inverse problem for a single snapshot can be solved using an ℓ_1-regularized least-squares optimization problem as earlier. Solving the near-field localization problem using sparse regression with or without Fresnel approximation for a single snapshot incurs the same complexity. Moreover, this approximation can even deteriorate the range estimation (for more details see [6]). However, when multiple snapshots are available, the structure of the Fresnel approximated array manifold matrix allows us to significantly reduce the computational complexity.

V. Two-step estimator with multiple snapshots

When there are multiple measurements available, we can stack (5) for the batch of T measurements into a matrix

$$\mathbf{Y} = \mathbf{A}(\bar{\boldsymbol{\theta}}, \bar{\mathbf{r}})\mathbf{X} + \mathbf{W} \quad (13)$$

where $\mathbf{Y} = [\mathbf{y}(t_1), \ldots, \mathbf{y}(t_T)] \in \mathbb{C}^{M\times T}$, and matrices $\mathbf{X}$ and $\mathbf{W}$ are defined similarly. An important point to be noted here is that the matrix $\mathbf{X}$ is sparse only spatially, and is generally not sparse in time. A straightforward approach would be to use a joint sparsity promoting ℓ_2/ℓ_1-norm regularization, or an ℓ_1-SVD [9] algorithm to solve the inverse problem. In this paper, we propose to reduce the involved computational complexity of 2D-gridding by solving an inverse problem of smaller dimensions in two-steps. We do this by exploiting the spatial cross-correlation between the symmetric sensors, and the fact that the structure of the Fresnel approximated model naturally decouples the DOA and range.

We now make the following assumptions[1]:

(a1) The source signals are mutually independent and are modeled as independent identically distributed (i.i.d.) complex circular random variables with zero mean and covariance matrix $\mathbb{E}_t\{\mathbf{s}(t)\mathbf{s}^H(t)\} = \mathrm{diag}(\sigma_{s,1}^2, \ldots, \sigma_{s,K}^2)$.

(a2) The noise is modeled as a zero-mean spatially white Gaussian process, and it is independent of the source signals. The noise covariance matrix is given by $\mathbb{E}_t\{\mathbf{w}(t)\mathbf{w}^H(t)\} = \sigma_w^2\mathbf{I}$.

Under assumptions (a1) and (a2), the spatial correlation between the m-th and n-th sensor can be written as

$$\begin{aligned} r_y(m,n) &= E_t\{y_m(t)y_n^*(t)\} \\ &= \sum_{k=1}^{K} \sigma_{s,k}^2 e^{j(m-n)\omega_k + j(m^2-n^2)\phi_k} + \sigma_w^2\delta(m-n) \end{aligned}$$

where $\delta(.)$ represents the Dirac function, $\mathbb{E}_t\{s_k(t)s_k^*(t)\} = \sigma_{s,k}^2$ denotes the signal power of the k-th source, and σ_w^2 is the noise variance. Notice that when $n = -m$ the spatial correlation is independent of the parameter ϕ_k [10], [11], and we arrive at

$$\begin{aligned} r_y(-m,m) &= E_t\{y_{-m}(t)y_m^*(t)\} \\ &= \sum_{k=1}^{K} \sigma_{s,k}^2 e^{-2m\omega_k j} + \sigma_w^2\delta(-2m). \end{aligned} \quad (14)$$

This means that by exploiting the cross-correlation between the symmetric sensors, we can transform the original 2D (DOA and range) estimation into a 1D (DOA) estimation. Stacking (14) for all the symmetric sensors, we can build a *virtual far-field* scenario:

$$\mathbf{r}_y = \mathbf{A}_\omega(\boldsymbol{\omega})\mathbf{r}_\mathbf{s} + \sigma_w^2\mathbf{e}, \quad (15)$$

where $\mathbf{r}_y = [r_y(-p,p), \ldots, r_y(0,0), \ldots, r_y(p,-p)]^T \in \mathbb{C}^{M\times 1}$, $\mathbf{r}_s = [\sigma_{s,1}^2, \ldots, \sigma_{s,K}^2]^T \in \mathbb{C}^{K\times 1}$, and $\mathbf{e} = [\mathbf{0}_p^T, 1, \mathbf{0}_p^T]^T \in \mathbb{C}^{M\times 1}$, and the corresponding

[1]These assumptions are required only for the two-step estimator discussed in Section V.

virtual array gain pattern for the k-th source denoted by $\mathbf{a}_\omega(\omega_k)$ can be expressed as $\mathbf{a}_\omega(\omega_k) = [e^{-j2p\omega_k}, \ldots, 1, \ldots, e^{j2p\omega_k}]^T \in \mathbb{C}^{M\times 1}$, with the array manifold $\mathbf{A}_\omega(\boldsymbol{\omega}) = [\mathbf{a}_\omega(\omega_1), \mathbf{a}_\omega(\omega_2), \ldots, \mathbf{a}_\omega(\omega_K)] \in \mathbb{C}^{M\times K}$. In practice, the vector $\mathbf{r}_y$ containing the statistical correlations is approximated using the measurements from (13).

A. Step-1: DOA estimation

As before, we can construct an overcomplete basis $\mathbf{A}_\omega$ but now with only N_θ columns corresponding to potential source directions of arrival (DOAs) using the sampling grid $\bar{\boldsymbol{\theta}}$, i.e.,

$$\mathbf{A}_\omega(\bar{\boldsymbol{\omega}}) = [\mathbf{a}_\omega(\bar{\omega}_1), \ldots, \mathbf{a}_\omega(\bar{\omega}_{N_\theta})] \in \mathbb{C}^{M\times N_\theta},$$

where $\bar{\omega}_n = -\frac{2\pi\delta}{\lambda}\sin(\bar{\theta}_n)$ for all $n \in \{1, \ldots, N_\theta\}$ as defined earlier. The signal is represented by an $N_\theta \times 1$ vector $\mathbf{u}(t)$, where every source can be found as a non-zero weight $u_n(t) = s_k(t)$ if source k comes from direction $\bar{\theta}_n$ for some k and is zero otherwise, i.e., the dominant peaks in $\mathbf{u}(t)$ correspond to the true source locations. The discrete grid-based model in the correlation domain is then given by

$$\mathbf{r}_y = \mathbf{A}_\omega(\bar{\boldsymbol{\omega}})\mathbf{u} + \sigma_w^2\mathbf{e}. \tag{16}$$

Note that the number of potential source DOAs N_θ will typically be much greater than the number of sensors M also in the correlation domain, and the model in (16) is still ill-posed. Hence, we solve for the unknown vector $\mathbf{u}$ using an ℓ_1-regularized LS minimization problem which is given by

$$\hat{\mathbf{u}} = \arg\min_{\mathbf{u}} \quad \|\mathbf{r}_y - \mathbf{A}_\omega(\bar{\boldsymbol{\omega}})\mathbf{u}\|_2^2 + \mu_1\|\mathbf{u}\|_1, \tag{17}$$

where μ_1 is the sparsity regulating parameter. Alternatively, when the noise variance σ_w^2 is known, the unknown vector $\mathbf{u}$ can be obtained by solving

$$\arg\min_{\mathbf{u}} \quad \|\mathbf{r}_y - \mathbf{A}_\omega(\bar{\boldsymbol{\omega}})\mathbf{u} - \sigma_w^2\mathbf{e}\|_2^2 + \mu_1\|\mathbf{u}\|_1.$$

B. Step-2: range estimation

Let $\hat{\boldsymbol{\theta}}$ be the estimated DOAs from step-1, and $\hat{K}$ denote the number of DOAs detected (i.e., $\hat{K} = \|\hat{\boldsymbol{u}}\|_0$). We now use the sampling grid $(\hat{\boldsymbol{\theta}}, \bar{\mathbf{r}})$ to form an overcomplete basis $\mathbf{A}(\hat{\boldsymbol{\theta}}, \bar{\mathbf{r}}) \in \mathbb{C}^{M\times \hat{K}N_r}$ to arrive at

$$\mathbf{Y} = \mathbf{A}(\hat{\boldsymbol{\theta}}, \bar{\mathbf{r}})\tilde{\mathbf{X}} + \mathbf{W}, \tag{18}$$

where $\tilde{\mathbf{X}}$ is obtained by removing some specific rows of the signal matrix $\mathbf{X}$. In order to solve the inverse problem in (18) we use the ℓ_1-SVD algorithm. Note that in step-2 for range estimation we do not use the Fresnel approximation anymore. For the sake of completeness, the ℓ_1-SVD algorithm in [9] is briefly summarized as follows.

Let $\mathbf{Y} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^H$ be the singular value decomposition (SVD) of the data matrix. Keep a reduced $M \times \hat{K}$ matrix $\mathbf{Y}_{\rm sv} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{D}_k = \mathbf{Y}\mathbf{V}\mathbf{D}_k$, where $\mathbf{D}_k = [\mathbf{I}_k, \mathbf{0}^T_{\hat{K}\times(T-\hat{K})}]$. The reduced data matrix contains most of the signal power, and forms the basis for the signal subspace. Similarly, let $\tilde{\mathbf{X}}_{\rm sv} = \tilde{\mathbf{X}}\mathbf{V}\mathbf{D}_k$ and $\mathbf{W}_{\rm sv} = \mathbf{W}\mathbf{V}\mathbf{D}_k$, to arrive at

$$\mathbf{Y}_{\rm sv} = \mathbf{A}(\hat{\boldsymbol{\theta}}, \bar{\mathbf{r}})\tilde{\mathbf{X}}_{\rm sv} + \mathbf{W}_{\rm sv}, \tag{19}$$

which can be expressed in vector form (column by column) as

$$\mathbf{y}_{\rm sv}(k) = \mathbf{A}(\hat{\boldsymbol{\theta}}, \bar{\mathbf{r}})\tilde{\mathbf{x}}_{\rm sv}(k) + \mathbf{w}_{\rm sv}(k), \text{ for } k = 1, \ldots, \hat{K}.$$

Here, each column corresponds to a signal subspace singular vector. The reduced data matrix is only spatially sparse, and not in terms of the singular vector index k. In order to take this effect into account, we use a different prior obtained by computing the ℓ_2-norm of the singular values of a particular spatial index of $\tilde{\mathbf{x}}_{\rm sv}(k)$, i.e., $\tilde{x}_{m,\rm sv}^{(\ell_2)} = \sqrt{\sum_{k=1}^{K}(\tilde{x}_{m,\rm sv}(k))^2}$, for $m \in [-p, p]$. Note that the ℓ_2-norm can be computed for all snapshots instead of only the signal subspace singular vectors, however, the former technique adds more computational complexity especially when $T \gg \hat{K}$ [9]. Now, we can find the range by minimizing

$$\|\mathbf{Y}_{\rm sv} - \mathbf{A}(\hat{\boldsymbol{\theta}}, \bar{\mathbf{r}})\tilde{\mathbf{X}}_{\rm sv}\|_F^2 + \mu_{\rm sv}\|\tilde{\mathbf{x}}_{\rm sv}^{(\ell_2)}\|_1,$$

where $\tilde{\mathbf{x}}_{\rm sv}^{(\ell_2)} = [\tilde{x}_{-p,\rm sv}^{(\ell_2)}, \ldots, \tilde{x}_{p,\rm sv}^{(\ell_2)}]^T$, and the parameter $\mu_{\rm sv}$ controls the spatial sparsity.

Remark 1 (Complexity reduction with multiple snapshots)**.** *Jointly estimating the DOA and range by applying the* ℓ_1*-SVD algorithm on the model* (13) *costs* $O((KN_\theta N_r)^3)$*. Using the proposed two-step estimator, the complexity is reduced significantly to* $O(KN_r^3)$ *(reduction by a factor of* $O(N_\theta^3)$*) with an additional complexity of solving the inverse problem in* (17)*, which costs for example,* $O(N_\theta\log(N_\theta))$ *using the iterative thresholding algorithm [12]. For a typical problem with* $N_r = 15$ *and* $N_\theta = 180$ *points on the grid, the complexity reduction is significant.*

Remark 2 (Array geometry)**.** *Any array (uniform or non-uniform) can be used to solve for the variables* $(\boldsymbol{\theta}, \mathbf{r})$ *based on the optimization problem in* (6)*. For the two-step approach, any symmetric array (uniform or non-uniform) can be used.*

VI. Simulations

We consider a ULA with $M = 15$ sensors placed such that the inter-sensor spacing is $\delta = \lambda/4$[2], where λ represents the wavelength of the narrowband source signals. For this array, the far-field distance is beyond $2D^2 = 24.5\lambda$, and any source within the range of 24.5λ from the array will be in the near field.

We compare the proposed algorithms with matched-filter beamforming [2] for a single snapshot scenario. For the multiple snapshot scenario we compare the performance also with the 2D-MUSIC algorithm [4]. The optimization problems in the proposed algorithms are solved using CVX [13]. The regularization parameter is chosen via cross-validation.

In Fig. 2, we illustrate the joint DOA and range estimation obtained by solving the sparse regression problem for a single snapshot scenario. We consider two sources at locations $(0°, 5\lambda)$ and $(10°, 10\lambda)$. The SNR is 20 dB with $T = 1$. As can be seen from the plots, high resolution can be achieved by using the sparse modeling framework as compared to the

[2]To avoid aliasing when the virtual far-field model in (15) is used.

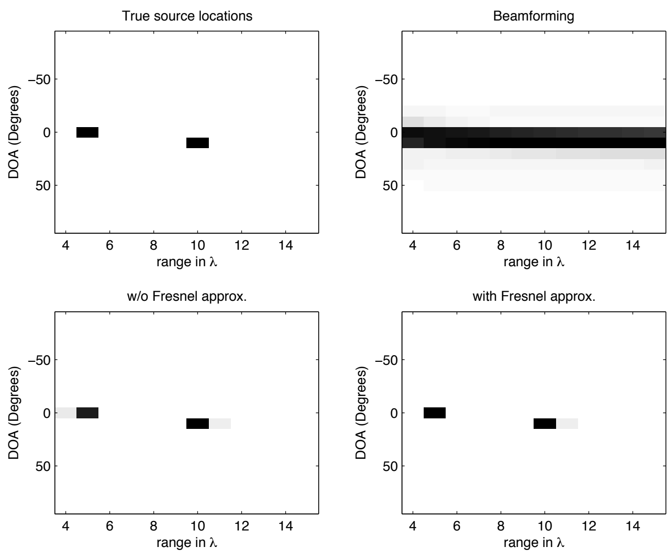

Fig. 2: Joint DOA and range estimation. Two near-field sources with DOAs: 0° and 10°, and ranges 5λ and 10λ. SNR = 20 dB, and $T = 1$. The sampling grid has a resolution of $\Delta\theta = 10^\circ$ and $\Delta r = \lambda$.

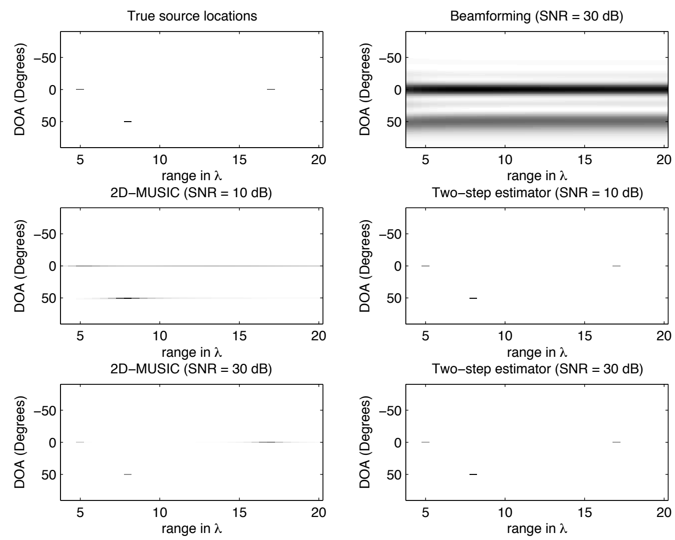

Fig. 3: Two-step estimator. Three near-field sources at locations: $(0^\circ, 5\lambda)$, $(0^\circ, 17\lambda)$, and $(50^\circ, 8\lambda)$, and $T = 200$. The sampling grid has a resolution of $\Delta\theta = 1^\circ$ and $\Delta r = 0.5\lambda$.

conventional beamforming technique. The performance loss with and without (w/o) the Fresnel approximation for the considered scenario is negligible.

In Fig. 3, we show the proposed two-step estimator for near-field source localization. We consider three near-field sources at locations $(0^\circ, 5\lambda)$, $(0^\circ, 17\lambda)$, and $(50^\circ, 8\lambda)$. The simulations are provided for SNRs of 10 dB and 30 dB with $T = 200$ snapshots. In the two-step estimator, we solve for the DOAs in the correlation domain in which the range parameters are naturally decoupled from the DOAs (due to the Fresnel approximation). In the second step, we solve for the range using the ℓ_1-SVD algorithm where we use the number of sources as $\hat{K}$ from step-1. Even though 2D-MUSIC can achieve a high resolution in the DOA domain, its performance deteriorates along the range domain especially when two sources share the same DOA. The two-step approach would further allow to increase the gridding resolution because of the involved smaller overcomplete dictionary as compared to the dictionary obtained from a 2D grid (cf. (5)).

VII. Conclusions

The classical near-field source localization problem is a non-linear (joint DOA and range) parameter estimation problem. The frequently used planar-wave assumption is no more valid for near-field sources as the wavefronts are spherical. Using the sparse representation framework, we transform the original non-linear problem into a linear ill-posed inverse problem. Based on the assumption that the spatial spectrum (i.e., the number of point sources) is sparse, we can localize the sources with a high resolution by solving an ℓ_1-regularized sparse regression. Additionally, when multiple snapshots are available and the sources are uncorrelated, the DOA and range parameters are naturally decoupled in the correlation domain. This enables us to solve two smaller dimension sparse regression problems instead of one higher dimension sparse regression problem which leads to a significant complexity reduction.

References

[1] P. Stoica and K.C. Sharman, "Maximum likelihood methods for direction-of-arrival estimation," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 38, no. 7, pp. 1132–1143, Jul. 1990.

[2] H. Krim and M. Viberg, "Two decades of array signal processing research: the parametric approach," *Signal Processing Magazine, IEEE*, vol. 13, no. 4, pp. 67–94, 1996.

[3] R.O. Schmidt, "Multiple emitter location and signal parameter estimation," *IEEE Transactions on Anennas and Propagation*, vol. AP-34, no. 3, pp. 276–280, Mar. 1986.

[4] Y-D. Huang and M. Barkat, "Near-field multiple source localization by passive sensor array," *IEEE Transactions on Anntennas and Propagation*, vol. 39, no. 7, pp. 968–974, Jul. 1991.

[5] N. Lv, J. Liu, Q. Wang, and J. Du, "An efficient second-order method of near-field source localizaiton and signal recovery," *IEEE 13th International Conference on Communication Technology (ICCT)*, pp. 780–784, Sep. 2011.

[6] A.L. Swindlehurst and T. Kailath, "Passive direction of arrival and range estimation for near-field sources," *IEEE Spec. Est. and Mod. Workshop*, pp. 123–128, 1988.

[7] K. Abed-Meraim, Y. Hua, and A. Belouchrani, "Second-order near-field source localization: algorithm and performance analysis," *Signal, System and Computers*, vol. 1, pp. 723–727, Nov. 1996.

[8] Y. Wu, Y. Dong, and G. Liao, "Joint estimation both range and doa of near-field source," *Journal of Electronics (China)*, vol. 21, no. 2, pp. 104–109, Mar. 2004.

[9] D. Malioutov, M. Cetin, and A.S. Willsky, "A sparse signal reconstruction perspective for source localization with sensor arrays," *IEEE Transctions on Signal Processing*, vol. 53, no. 8, Aug. 2005.

[10] K. Abed-Meraim, Y. Hua, and A. Belouchrani, "Second-order near-field source localization: algorithm and performance analysis," in *In proc. of the Asilomar Conference on Signals, Systems and Computers (ASILOMAR)*, Nov 1996, vol. 1, pp. 723–727 vol.1.

[11] Yinghui Zhao, A. Dinstel, M.R. Azimi-Sadjadi, and N. Wachowski, "Localization of near-field sources in sonar data using the sparse representation framework," in *OCEANS 2011*, Sept 2011, pp. 1–6.

[12] A. Maleki and D.L. Donoho, "Optimally tuned iterative reconstruction algorithms for compressed sensing," *IEEE J. Sel. Topics in Sig. Proc.*, vol. 4, no. 2, pp. 330–341, 2010.

[13] [Online], "CVX: Matlab software for disciplined convex programming, version 2.0 beta," http://cvxr.com/cvx, Sep. 2012.