A least-squares implementation of the field integrated method to solve time domain electromagnetic problems

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***Abstract*—This paper presents the application of the least-squares field integrated method based on hybrid linear finite elements in time-domain electromagnetic (EM) problems with high contrast interfaces. The method proposes the use of edge based linear finite elements over nodal elements and edge elements of Whitney form. It shows how the equations have to be accommodated to yield a correct solution. It proposes a general strategy to combine edge linear finite elements and nodal linear finite elements. Numerical experiments show that the resulting algorithms are stable and achieve high quality field interpolation even in the presence of very high contrasts.**

***Index Terms*—least-squares, Field Integrated Equations, time domain computational electromagnetism, hybrid finite element, high contrast**

I. INTRODUCTION

In strongly heterogeneous media, the constitutive parameters can jump by large amounts upon crossing the material interfaces. On a global scale, the EM field components are, therefore, not differentiable and Maxwell's equations in differential form cannot be used: one has to resort to the original integral form of the EM field relations as the basis for the computational method. The appropriate integral form is provided by the classical interrelations between the curl of the electric/magnetic field strength along a closed curve and the time rate of change of the magnetic/electric flux passing through a surface with the circulation loop as boundary. For these to hold, only integrability of the field is needed, which condition we impose in accordance with the physical condition of boundedness of the field quantities. To satisfy the constitutive relations (that are representative of physical volume effects), a fitting continuation of the boundary representations of the field components of an element into its interior is needed. We construct a consistent algorithm that meets all of these requirements, using a simplicial geometrical discretization combined with piecewise linear representation of the electric and magnetic field components along the edges of the elements, piecewise linear extrapolations into the interior of the elements and taking constant values of the constitutive coefficients in these interiors. Furthermore, we use NETGEN[8] to discretize the computational domain with a 2D(triangular)/3D(tetrahedra) mesh. We use simple boundary conditions (PEC, PMC) to truncate the computational domain. After properly assembling the local matrices, we obtain a symmetric positive definite system of algebraic equations, which we solve with a preconditioned iterative method. We test the accuracy of the method by implementing the four domain problem treated analytically in [4] in the time domain using our method. This experiment also documents the stability of our approach.

II. FIELD REPRESENTATION

In this paper we consider a 2D situation (the method applies perfectly well to 3D but the main effects can be demonstrated effectively in 2D), we use 'finite elements' consisting of triangles, and approximate the fields by linear interpolation inside the elements. Due to the nature of the interface conditions, a straight-forward application of the linear expansion function across boundaries would lead to large numerical error or excessive mesh refinement. Applying these interfaces conditions as constraints would result in semi-positive definite system matrices which are difficult to solve (see [7], [1]). It is advantageous to take them directly into account when discretizing the field quantities. The key point we propose is to approximate the field quantities, which are known to be continuous, with nodal linear finite elements, and the discontinuous ones are approximated with edge based finite elements.

A. Geometrical quantities

Before introducing the linear expansion functions (shape functions), we define a few geometrical quantities (see Fig. 1). Let $\Delta(r_i, r_j, r_k)$ or $\Delta(i, j, k)$ be the triangle delimited by three vertexes with coordinates r_i, r_j, r_k; $|\Delta(i, j, k)|$ the area in triangle $\Delta(i, j, k)$; r_i the spatial coordinate of the vertex i; $e_{ij} = \frac{r_j - r_i}{|r_j - r_i|}$ the unit vector pointing from vertex i to j; $a_k = i_z \times e_{ij}$ the normal unit vector perpendicular to the edge delimited by vertex i and j; $\phi_i(r) = \frac{|\Delta(r, r_j, r_k)|}{|\Delta(r_i, r_j, r_k)|}, \forall r \in \Delta(r_i, r_j, r_k)$ the linear shape function, which, by definition, is equal to 1 on vertex i and 0 on other vertexes in $\Delta(r_i, r_j, r_k)$.

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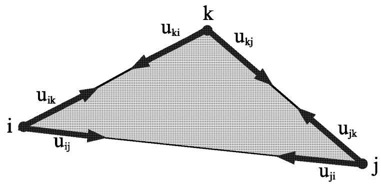

Fig. 1. Degrees of freedom on a two-dimensional consistently linear finite element.

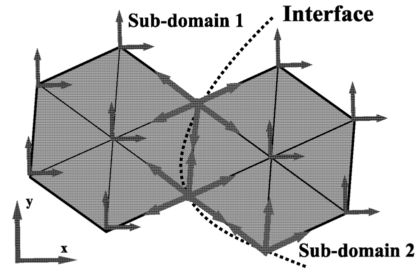

Fig. 2. The allocation of the nodal- and edge- linear finite elements.

B. Nodal linear finite element

In a homogeneous sub-domain, any field quantity - say v - is continuous. To preserve continuity, the field quantity is best approximated by continuous linear splines interpolating between nodes (so called 'nodal elements'). We use linear expansion functions to construct the local linear interpolation $\mathbf{v}$ of v (we use boldface for approximated quantities)

$$\mathbf{v}(r) = \sum_{l\in(i,j,k)} v(l)\phi_l(r). \tag{1}$$

Both the tangential component and normal component of $\mathbf{v}$ are continuous across the interface. For quantities belonging to $\mathcal{H}^1(\Omega) = \{v \in L^2(\Omega); v' \in L^2(\Omega)\}$, the approximation error is of the order $O(h^2)$ (where h denotes the mesh size).

C. Edge based consistently linear finite elements

For vectorial quantities that are (partially) discontinuous across interfaces, we have to proceed differently. To keep the tangential components of a vectorial field quantity $\vec{u}$ continuous across a interface, we use only the well defined continuous components (u_{lh} where $l, h \in (i,j,k), l{\neq}h$, see Fig. 1) on vertexes to construct a linear interpolation over the triangle $\Delta(i,j,k)$ on which edge it is defined. Let $\vec{u}(i)$ specify the two-dimensional field quantity on vertex i. It can be fully expressed in terms of the tangential components on the incident edges:

$$\vec{u}(i) = u_{ij}\frac{a_j}{e_{ij}{\bullet}a_j} + u_{ik}\frac{a_k}{e_{ik}{\bullet}a_k} = \sum_{l\in(i,j,k),l\neq i} u_{il}\frac{a_l}{e_{il}{\bullet}a_l}. \tag{2}$$

With $\vec{u}$ defined on each vertex, the linear shape function $\phi_l(r)$ can be used to construct the local linear interpolation $\vec{\mathbf{u}}(r)$ of $\vec{u}(r), r \in \Delta(i,j,k)$, as follows:

$$\begin{aligned}\vec{\mathbf{u}}(r) &= \sum_{l\in(i,j,k)} \vec{u}(l)\phi_l(r) \\ &= \sum_{l\in(i,j,k)} \sum_{h\in(i,j,k),h\neq l} u_{lh}\frac{a_h}{e_{lh}{\bullet}a_h}\phi_l(r). \end{aligned}\tag{3}$$

Unlike the Whitney-1 elements, the consistently linear element does not satisfy the divergence free condition. Therefore we need to enforce this condition explicitly. On the other hand, for quantities belonging to $\mathcal{H}(curl;\Omega) = \{\vec{v} \in [L^2(\Omega)]^{n_d} : \nabla\times\vec{v} \in [L^2(\Omega)]^{n_d}\}$, the consistently linear expansion has a local approximation error of the order $O(h^2)$ instead of $O(h)$ for Whitney edge elements. The consistently linear expansion function was used first by Mur and de Hoop in [2].

D. Combination of node- and edge- finite element expansions

To preserve the continuity properties of field quantities without introducing too many unnecessary unknowns, we use edge based consistently linear finite elements only on interfaces between different materials and node linear finite elements in homogeneous subdomains (see Fig. 2). Hereinafter, we refer to this combination as hybrid linear finite elements.

III. Two dimensional electromagnetic problem: the perpendicular polarization case

In this section, we show how to utilize the hybrid linear finite elements to solve 2D electromagnetic field problems.

A. Maxwell's equations

In the computation domain Ω with boundary $\partial\Omega$, given a (sufficiently smooth and small) surface S with boundary ∂S, Maxwell's equations in the surface integrated form are

$$\oint_{\partial S} H{\bullet}\,dl = \int_S \{\partial_t D + \sigma^e E + J^{imp}\}{\bullet}\,ds, \tag{4}$$

$$\oint_{\partial S} E{\bullet}\,dl = -\int_S \{\partial_t B + \sigma^m H + K^{imp}\}{\bullet}\,ds. \tag{5}$$

where $\sigma^e(x_r)$ is the electric conductivity, $\sigma^m(x_r)$ is the magnetic conductivity. Let $K^{tot} = \partial_t B + \sigma^m H + K^{imp}$, $J^{tot} = \partial_t D + \sigma^e E + J^{imp}$, consider a volume V with its closed surface ∂V, The compatibility relations in their integrated form are:

$$\oint_{\partial V} J^{tot}{\bullet}\,ds = 0, \tag{6}$$

$$\oint_{\partial V} K^{tot}{\bullet}\,ds = 0. \tag{7}$$

The computational domain is truncated by PEC and PMC boundary conditions:

$$n\times E = 0 \text{ on } \partial\Omega_1, \quad n{\bullet}K^{tot} = 0 \text{ on } \partial\Omega_1 \text{ (PEC boundary)},$$
$$n\times H = 0 \text{ on } \partial\Omega_2, \quad n{\bullet}J^{tot} = 0 \text{ on } \partial\Omega_2 \text{ (PMC boundary)}.$$

where $\partial\Omega_1 \cap \partial\Omega_2 = \phi$, $\partial\Omega_1 \cup \partial\Omega_2 = \partial\Omega$; Last but not the least, the interface conditions are:

$$[n{\bullet}K^{tot}] = 0 \text{ on } \Gamma_i, \quad [n\times H] = 0 \text{ on } \Gamma_i,$$
$$[n{\bullet}J^{tot}] = 0 \text{ on } \Gamma_i, \quad [n\times E] = 0 \text{ on } \Gamma_i.$$

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where $[A] = \lim A(\Gamma^+) - \lim A(\Gamma^-)$ denotes the jump of a quantity A across the material interface Γ. The constitutive relations are:

$$D(x_r, t) = \varepsilon(x_r)E(x_r, t), \quad B(x_r, t) = \mu(x_r)H(x_r, t).$$

where $\varepsilon(x_r)$ is the permittivity, $\mu(x_r)$ the permeability.

B. Two-dimensional electromagnetic problems and spatial-temporal- discretization

The 2D problem is characterized by invariance in the z direction. And we consider perpendicular polarized field only. In this case, the electric field strength is interpolated with nodal linear finite elements, since this field is always tangential to the material interfaces. The magnetic field strength, however, is approximated by hybrid linear finite elements as mentioned in Section II-D. This means that the magnetic field strength is interpolated by edge based linear finite element at the material interfaces and by nodal linear finite element in homogeneous sub-domains. Our discretization procedure is similar to that in [5], except that the discretized Maxwell's equations are derived there for static problems, while we work with the full Maxwell system in the time-domain. To implement a time stepping scheme for the spatially discretized Maxwell's equations, we introduce the time instance $t_i = i\delta t$, where $\delta t > 0$ is the time step, and integrate Maxwell's equations from $t = t_{i-1}$ to $t = t_i$. All integrals that can not be computed analytically are discretized using the trapezoidal rule. To maintain accuracy in the time-domain and avoid computing too many unnecessary time-steps, we choose the time-step δt corresponding to a CFL number between 1 and 2 for the smallest element (see [3]).

C. Normalization

Before formulating the system of equations, it is important to normalize these equations so that the system has better spectral properties. Let L be a problem related reference length. We normalize the spatial coordinate, time coordinate, field quantities, EM sources and matter parameters as follows:

$$\widehat{x} = \frac{x}{L}, \;\; \widehat{t} = \frac{c_0 t}{L}, \;\; \widehat{E}(\widehat{x}, \widehat{t}) = E(L\widehat{x}, \frac{L\widehat{t}}{c_0}),$$

$$\widehat{H}(\widehat{x}, \widehat{t}) = \sqrt{\frac{\mu_0}{\varepsilon_0}} H(L\widehat{x}, \frac{L\widehat{t}}{c_0}), \;\; \widehat{J}^{imp}(\widehat{x}, \widehat{t}) = L\sqrt{\frac{\mu_0}{\varepsilon_0}} J^{imp}(L\widehat{x}, \frac{L\widehat{t}}{c_0}),$$

$$K^{imp}(\widehat{x}, \widehat{t}) = LK^{imp}(L\widehat{x}, \frac{L\widehat{t}}{c_0}), \;\; \widehat{\sigma}^e(\widehat{x}) = L\sqrt{\frac{\mu_0}{\varepsilon_0}}\sigma^e(L\widehat{x}),$$

$$\widehat{\sigma}^m(\widehat{x}) = L\sqrt{\frac{\varepsilon_0}{\mu_0}}\sigma^m(L\widehat{x}), \;\; \widehat{\varepsilon}(\widehat{x}) = \varepsilon(L\widehat{x}), \;\; \widehat{\mu}(\widehat{x}) = \mu(L\widehat{x}).$$

We implement the least-squares field integrated method using the normalized Maxwell system.

D. The algebraic linear system and iterative solution methods

After the spatial and the temporal dicretization, a linear system of algebraic equations is to be solved. In view of very large system matrices are involved, an efficient and fast linear system solver is needed.

1) The linear system: After the linear system of integral equations has been discretized in the space domain, we express the time update by using the trapezoidal rule (this is a dicretization in the time domain that is consistent with the space domain discretization, and also preserves system stability - a conservative system will stay conservative, and a dissipative system will stay dissipative.) After all discretizations have been executed, the system has been reduced to a recursive set of algebraic equations summarized by the formula:

$$A_2 u_i = -A_1 u_{i-1} + G_i. \tag{8}$$

where u_{i-1} collects the solution of the time instant t_{i-1}, $u_i = \begin{bmatrix} H_i & E_i \end{bmatrix}^T$ collects the solution at the current time instant, and G_i collects the source and boundary terms. Finally, u_0 collects the initial field strength. This system is actually (slightly) over-determined due to the additional formulation of the divergence equations. We use a direct weighted least squares solution method as is customary in linear algebra when an over-determined system is encountered. This can always be done in a numerically stable way (the compatibility equations may be provided with weighting factors to tune their influence, if desired). An alternative approach would consist in expressing the original set of integral equations in discretized form as a collection of equations that must be solved in least squares sense, given a set of weights that tune the relative importance of each contributing equation. We prefer the direct method just discussed for its simplicity. It can be argued that this approach actually achieves the same objective.

2) The preconditioned CG method: the symmetric positive definite system can now be solved with any preconditioned Krylov space iterative solver. The preconditioner we use is the incomplete Cholesky factorization (IC) with dropping threshold 10^{-3} ($IC(10^{-3})$). However, direct application of $IC(10^{-3})$ on the matrix A_2 introduces a lot of fill-ins in the incomplete Cholesky factor. Applying the approximate symmetric minimum degree ordering [6] to the matrix A_2 significantly reduces the fill-ins in the incomplete Cholesky factor . After this pre-processing step, preconditioned Krylov space iterative solvers can be used to solve the discrete Maxwell's system (which is symmetric positive definite). The solution method normally takes less than 10 iterations to reach an accuracy 10^{-6}. Fewer iterations are needed if the solution of the previous time instant is taken for the initial guess of the current time instant.

IV. Numerical experiment

We test our method on a special example for which an analytic solution is available. The solution is a 'steady state' solution at a single frequency, containing a source term that continuously injects current. Since we look for a time-domain solution, we use the steady solution at $t = 0$ as initial state, and then start integration in the time-domain. Our solution should follow the actual theoretical steady state solution faithfully. We show that the solution stays stable and the error divergence of our method is much lower than with the classical nodal method using the same coarsely discretized mesh. The configuration is

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TABLE I
CONFIGURATION OF THE FOUR SUB-DOMAINS

Ω_i	Definition of sub-domains	μ_r	σ^m	ϵ_r	σ^e
Ω_1	$0 \le x < 0.5$ $0 \le y < 0.5$	1.25	0	1.0	0
Ω_2	$0.5 \le x \le 1$ $0 \le y < 0.5$	2.5	0	1.0	0
Ω_3	$0 \le x < 0.5$ $0.5 \le y \le 1$	1	0	1.0	0
Ω_4	$0.5 \le x \le 1$ $0.5 \le y \le 1$	1000	0	1.0	0

a square domain $\Omega = \{0 \le x \le 1, 0 \le y \le 1\}$ consisting of four sub-domains $\Omega_i, \{i = 1, 2, 3, 4\}$ with different medium properties (See Table I). Let

$$h(t) = \frac{\sigma^m}{(\sigma^m)^2 + \mu^2\omega^2}\cos(\omega t) + \frac{\mu\omega}{(\sigma^m)^2 + \mu^2\omega^2}\sin(\omega t),$$
$$g(t) = \sigma^e\cos(\omega t) - \epsilon\omega\sin(\omega t).$$

and the source density distributions is given by:

$$J_z^{imp} = [-2\pi^2 h(t) - g(t)]\sin(\pi x)\sin(\pi y), \quad (9)$$
$$K_x^{imp} = 0, \quad (10)$$
$$K_y^{imp} = 0. \quad (11)$$

The exact field strengths are then:

$$E_z = \sin(\pi x)\sin(\pi y)\cos(\omega t), \quad (12)$$
$$H_x = -\pi h(t)\sin(\pi x)\cos(\pi y), \quad (13)$$
$$H_y = \pi h(t)\cos(\pi x)\sin(\pi y). \quad (14)$$

The angular frequency ω is chosen to be $\pi \times 10^9 \mathrm{rad}/s$. The same test configuration in the frequency domain was used by Jorna in [4]. Let the Root Mean Square Error for the field strength F in Ω at time t be:

$$RMSE(F,t) = \left(\frac{\int_\Omega |F(r,t) - F_{exact}(r,t)|^2}{\int_\Omega |\max_{t\in[0,T]} F_{exact}(r,t)|^2}\right)^{\frac{1}{2}} \times 100\%. \quad (15)$$

With an interface conforming mesh (consists of 289 points and 512 triangular elements), the $RMSE$s in Ω are plotted in Fig. 3. Note that all $RMSE$s start with 0 at $t = 0$ because the exact solution is taken as the initial value. Experiments have shown that the field strengths computed with mixed elements stay stable and reasonably accurate for this coarse mesh; while the field strengths computed with nodal elements along are unstable and inaccurate. Note that $\mu_r = 1000$ in Ω_4, since the mesh we used is not fine, any method would produce large numerical errors in Ω_4. However, with our method, the error in Ω_4 does not contaminate the numerical solutions in other regions. A snapshot of the field strength computed by hybrid finite elements is shown in Fig. 4.

V. CONCLUSION

The Least-squares field integrated method based on hybrid linear finite elements holds considerable promise to model electromagnetic effects in integrated circuits, where high contrasts between different types of materials is the rule and complex structures are present.

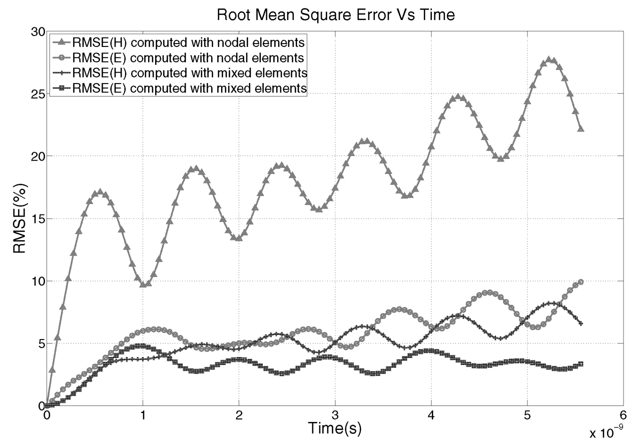

Fig. 3. The root mean square error in the electric field strength (E) and the magnetic field strength (H) in $\Omega \times [0, T]$. 100 time steps have been computed and plotted. The RMSE Vs time plot of the electromagnetic field strengths computed with field integrated method based hybrid linear finite elements indicates that the method is accurate and stable.

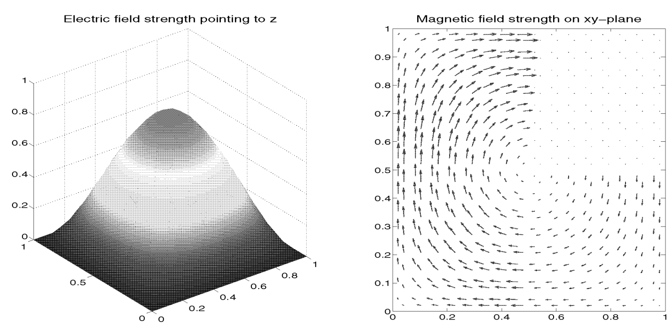

Fig. 4. The snapshot of the electric and magnetic field strengths at $t = 1.667 \times 10^{-10}(s)$ computed with the least-squares field integrated method based on hybrid linear finite elements. The upper right sub-domain is Ω_4 ($\mu_r = 1000$). Errors in Ω_4 do not contaminate the solutions in other regions.

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