

# Sparsity-Enforcing Sensor Selection for DOA Estimation

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**Abstract**—The sensitivity of direction-of-arrival (DOA) estimation to different array geometries motivates the design of optimal sensor constellations. We propose a framework for array geometry design for a linear array with fixed aperture and fixed inter-element spacing, where the array geometry design is formulated as a sensor selection problem. The sensor selection is performed such that it achieves a desired Cramér-Rao bound (CRB) for estimating the DOA of a single source. The non-uniformity of the sensor selection typically results in sidelobes. These sidelobes are suppressed in a specified angular sector again via sensor selection. The aforementioned problems are jointly casted as a semidefinite programming (SDP) problem which can be efficiently solved in polynomial time. Simulations exhibit the trade-offs among the number of selected sensors, sidelobe minimization, and CRB of the DOA estimates.

## I. INTRODUCTION

The optimal selection of sensors in an array paves the way for creating a trade-off between the performance and cost of array design for direction-of-arrival (DOA) estimation. Practical applications can be off-line antenna selection in the context of station planning for radio astronomy, as well as weather monitoring applications in which resources like the number of sensors, and maximum allowable aperture are already known, and are generally limited.

A plethora of excellent techniques has already been proposed since the early eighties. Several mathematical tools like convex optimization and biologically-inspired algorithms have been extensively used by both array signal processing and antenna research communities, to solve the preceding problem. In [1], for example, an optimal sensor placement for a fixed aperture has been proposed which minimizes the Cramér-Rao bound (CRB) using a genetic algorithm, and this results in an optimal array with clustered sensors. Another well-known closely related topic to DOA estimation in array processing is beamforming. Sensor array optimization via convex optimization in the context of beamforming has been proposed in [2], [3]. In [4], the Weiss-Weinstein bound (WWB) has been used to optimize the array geometry. In [5], the array geometry is optimized by minimizing the Bayesian CRB (BCRB), where a prior probability density function (PDF) of the source DOA is available. A method for optimizing the array geometry by minimizing a modified beampattern has been proposed in [7]. Recently, a technique for sensor subset selection in the context

of localization has been proposed where the sensor placement problem subject to performance constraints has been posed as the design of a sparse selection vector [6].

Inspired by [6], we propose in this work a *sensor selection* methodology for array design where we assume that the aperture width is fixed and the available sensor locations are known in advance. We propose a methodology to design a *sparse sensor selection* vector such that the designed array jointly achieves a desired performance (CRB) for a given DOA and minimizes the sidelobes upto a prescribed level in a specified angular sector. The difference of the proposed method with most of the existing approaches is avoiding the usage of the second-order statistics of the received signal at different sensors. In contrast, we directly select the sensors by formulating an  $\ell_1$ -norm optimization problem constrained by the performance requirements, and this convex optimization problem can be efficiently solved in polynomial time. Enforcing sparsity in sensor selection improves the cost-efficiency (hardware, processing, etc.) of the array design achieving some desired performance.

The outline of the paper is as follows. In section II, we discuss the data model for single source DOA estimation, the CRB of the DOA estimate, and its relation with ambiguity and beamwidth. In section III, we formulate the proposed optimization problem. In section IV, the simulation results are presented.

## II. PRELIMINARIES

### A. Data model

Let us consider a far-field narrowband source  $s(t)$  with  $t$  denoting the discrete time index characterized by a DOA denoted by  $\theta$ . Let  $a_i(\theta)$  be the gain pattern of the  $i$ -th sensor located at position  $p_i$  (a one-dimensional linear array is used). For a linear array with  $M$  sensors, the received signal can be collected in an  $M \times 1$  vector  $\mathbf{r}(t)$  which leads to the well-known data model [1], [5], [8]

$$\mathbf{r}(t) = \mathbf{a}(\theta)s(t) + \mathbf{n}(t), \quad t = 1, \dots, N, \quad (1)$$

where  $N$  denotes the number of snapshots, and  $\mathbf{n}(t)$  indicates an  $M \times 1$  additive noise vector. Assuming the centroid  $p_c = \frac{1}{M} \sum_{i=1}^M p_i$  of the array as its phase reference, the array steering vector is given by  $\mathbf{a}(\theta) = [e^{j \frac{2\pi}{\lambda} (p_1 - p_c) \sin \theta}, \dots, e^{j \frac{2\pi}{\lambda} (p_M - p_c) \sin \theta}]^T \in \mathbb{C}^{M \times 1}$  with  $\lambda$  being the wavelength of the signal. It is assumed that the signal

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and the noise are mutually uncorrelated, and are respectively modeled as  $s(t) \sim \mathcal{CN}(0, \sigma_s^2)$  and  $\mathbf{n}(t) \sim \mathcal{CN}(0, \sigma_n^2 \mathbf{I}_M)$ .

### B. CRB for single source DOA estimation

One of the most important performance metrics for DOA estimation is the CRB, which is a lower bound on the variance of any unbiased estimator. The reason behind selecting the CRB as a performance criterion can be justified by its mathematically tractable structure and relation with the beamwidth of the array. The analytical formulation of the CRB for DOA estimation for any number of sources, and for any arbitrary array geometry is derived in [8]. In [9], the CRB for a one-dimensional linear array is explicitly illustrated. It is also shown in [10], that the array geometry dependent terms in the expression of the CRB are independent of the narrow-band/wideband assumption on the signal. For a deterministic DOA  $\theta$ , the expression of the CRB denoted by  $C(\theta)$  (or the Fisher information denoted by  $F(\theta)$ ) for single source DOA estimation is given by [9]

$$F(\theta) = C^{-1}(\theta) = 2N\gamma \left( \frac{2\pi}{\lambda} \cos \theta \right)^2 \sum_{i=1}^M (p_i - p_c)^2, \quad (2)$$

where the term  $\gamma$  is related to the signal-to-noise ratio (SNR). It is clear from (2), that the dominant factors responsible for the estimation performance for a fixed  $\theta$  are  $\lambda$ ,  $\gamma$ ,  $N$ ,  $M$ , and the *array topology* [9]. Note that we can also rewrite (2) as

$$\begin{aligned} F(\theta) &= \kappa \cos^2 \theta \sum_{i=1}^M \left( p_i - \frac{1}{M} \sum_{i=1}^M p_i \right)^2 \\ &= \frac{\kappa}{M} \cos^2 \theta \sum_{i=1}^{M-1} \sum_{j=i+1}^M (p_i - p_j)^2, \end{aligned} \quad (3)$$

where we have introduced a constant  $\kappa := 2N\gamma \left( \frac{2\pi}{\lambda} \right)^2$ .

### C. Relationship of CRB with ambiguity and beamwidth

The unambiguous estimation of the DOA of the source is one of the main performance criteria of any array geometry. For any small change in DOA denoted by  $\Delta\theta$ , the function  $f(\Delta\theta) = \|\mathbf{a}(\theta) - \mathbf{a}(\theta + \Delta\theta)\|_2^2$  should be very high. The second-order term from the Taylor series expansion of  $f(\Delta\theta)$  around  $\Delta\theta = 0$  is proportional to [11]

$$f(\Delta\theta) \propto (\Delta\theta)^2 \cos^2 \theta \sum_{i=1}^M (p_i - p_c)^2.$$

The right-hand side of the above expression directly relates to the Fisher information for DOA estimation. So it can be said that maximizing the Fisher information will lead to a more unambiguous solution.

From the analysis of [10], it can be inferred that the half-power beamwidth of the array is approximately proportional to the CRB. This property directly relates to the estimation performance. However, the CRB does not take care of the sidelobes, which substantially degrades the estimation performance in low-SNR scenarios. In the following section, we will propose a countermeasure for that.

## III. OPTIMIZATION PROBLEM

In this section, we will focus on optimizing the sensor positions in the array. We could do this by minimizing the CRB in (3) over the  $p_i$ 's, given a fixed aperture  $A = p_M - p_1$ . However, since this is a difficult optimization problem, we prefer to use another approach. Starting from a uniform linear array (ULA) with half-wavelength spacing and aperture  $A = M\lambda/2$ , i.e.,  $p_i = b + i\lambda/2$  with  $b$  some arbitrary offset and  $\lambda$  the wavelength of the signal, we try to select the best subset of sensors from this ULA of  $M$  sensors using a *selection* vector  $\mathbf{w} = [w_1, \dots, w_M]^T \in \{0, 1\}^M$ , where  $w_i = 1(0)$  indicates that the  $i$ -th sensor in the original array of  $M$  sensors is (not) selected. This way, the aperture of the optimal array with a subset of sensors will be the same as the original array of  $M$  sensors. The sensor selection should minimize the number of active sensors while guaranteeing some desired CRB. For this purpose, introducing the selection-vector  $\mathbf{w}$  and using  $p_i = b + i\lambda/2$  in (3), we get

$$F(\mathbf{w}, \theta) = \frac{\kappa'}{\sum_{i=1}^M w_i} \cos^2 \theta \sum_{i=1}^{M-1} \sum_{j=i+1}^M w_i w_j (i-j)^2, \quad (4)$$

where  $\kappa' = \kappa\lambda^2/4$ . Note that we replace  $M$  in the denominator of (3) with the cardinality of the active sensor set  $\sum_{i=1}^M w_i = \|\mathbf{w}\|_0$  (for  $\mathbf{w} \in \{0, 1\}^M$ ). Here, the  $\ell_0$ -norm refers to the number of non-zero entries in  $\mathbf{w}$ . We can further simplify (4) to

$$F(\mathbf{w}, \theta) = \frac{\kappa'}{\|\mathbf{w}\|_0} \cos^2 \theta \sum_{i=1}^{M-1} \sum_{j=i+1}^M v_{ij} d_{ij}, \quad (5)$$

where  $v_{ij} = w_i w_j$  is related to the selection parameter and  $d_{ij} = (i-j)^2$  is the squared mutual difference between the  $i$ -th and the  $j$ -th sensor. In other words, we introduce a selection parameter  $v_{ij}$  to activate some differences between sensors. Here,  $v_{ij} = 1(0)$  with  $i < j$ , indicates that the corresponding difference term  $d_{ij}$  is (not) selected. Let us define a vector  $\mathbf{v} = [v_{12}, \dots, v_{(M-1)M}]^T \in \{0, 1\}^D$  with  $D = M(M-1)/2$ , and a vector of all possible *squared mutual differences*  $\mathbf{d} = [d_{12}, \dots, d_{(M-1)M}]^T \in \mathbb{R}^D$ . A suitable mapping between the entries of the vectors  $\mathbf{v}$  and  $\mathbf{w}$  is given by  $\mathbf{v} = \mathbf{w} \boxtimes \mathbf{w}$ , where  $\boxtimes$  is a Kronecker-like vector product defined as the element-wise multiplication between all possible combinations of elements in  $\mathbf{w}$  or in other words  $v_{ij} = w_i w_j$  for  $i < j$ . This can also be written as  $\mathbf{V} = \mathbf{w} \mathbf{w}^T$ , where  $[\mathbf{V}]_{ij} = [\mathbf{V}]_{ji} = v_{ij}$ ,  $\forall i = 1, \dots, M-1$ ,  $\forall j = i+1, \dots, M$  (off-diagonal elements of  $\mathbf{V}$ ).

### A. Selection based on the CRB constraint

The sensor subset selection can now be formulated as the design of a *sparse selection vector* with a constraint on the performance. More specifically, as a performance metric we lower bound the Fisher information, such that the CRB is upper bounded. The corresponding optimization problem can

be written as

$$(\hat{\mathbf{V}}, \hat{\mathbf{w}}) = \arg \min_{\mathbf{V} \in \mathbb{R}^{M \times M}, \mathbf{w} \in \{0,1\}^M} \|\mathbf{w}\|_0 \quad (6a)$$

$$\text{s.t.} \quad \frac{\kappa'}{\|\mathbf{w}\|_0} \cos^2(\theta) \mathbf{v}^T \mathbf{d} \geq \alpha, \quad (6b)$$

$$\mathbf{V} = \mathbf{w}\mathbf{w}^T, \quad (6c)$$

$$[\mathbf{V}]_{ij} = [\mathbf{V}]_{ji} = v_{ij}, \quad i = 1, \dots, M-1, \\ j = i+1, \dots, M. \quad (6d)$$

Here,  $\alpha$  is the threshold on the Fisher information allowing us to reduce the number of sensors. It can for instance be calculated by scaling  $F(\theta)$  in (3), i.e., the Fisher information of the full array. A tighter performance bound can be imposed by increasing the value of  $\alpha$ .

The optimization problem in (6) is non-convex due to the following reasons:  $\ell_0$ -norm cost function,  $\ell_0$ -norm in the denominator of (6b), and the rank-1 constraint in (6c).

We next use some standard convex relaxations, to simplify the problem, and the relaxed convex problem can then be written as the following SDP problem

$$(\hat{\mathbf{V}}_1, \hat{\mathbf{w}}_1) = \arg \min_{\mathbf{V} \in \mathbb{R}^{M \times M}, \mathbf{w} \in \mathbb{R}^M} \|\mathbf{w}\|_1 \quad (7a)$$

$$\text{s.t.} \quad \frac{\kappa'}{\mu} \cos^2(\theta) \mathbf{v}^T \mathbf{d} \geq \alpha, \quad (7b)$$

$$\begin{bmatrix} \mathbf{V} & \mathbf{w} \\ \mathbf{w}^T & 1 \end{bmatrix} \succeq 0, \quad (7c)$$

$$[\mathbf{V}]_{ij} = [\mathbf{V}]_{ji} = v_{ij}, \quad i = 1, \dots, M-1, \\ j = i+1, \dots, M, \\ [\mathbf{V}]_{ii} = w_i, \quad i = 1, \dots, M, \quad (7d)$$

where the  $\ell_1$ -norm defined as  $\|\mathbf{w}\|_1 = \sum_{i=1}^M w_i$  is a convex relaxation for  $\|\mathbf{w}\|_0$  in (6a), and we use  $\mu$  as a guess for  $\|\mathbf{w}\|_0$  in (6b). Taking  $\mu = M$  we provide a weaker, yet, sufficient constraint for the Fisher information (we provide more details on refining  $\mu$  in the next section). (7c) is the relaxation for the rank-1 constraint in (6c), and (7c) and (7d) jointly yield the Lagrangian relaxation for  $\{0,1\}^M$  and are equivalent to  $0 \leq w_i \leq 1$  for  $i = 1, \dots, M$ .

Note that we solve (7) only for a single  $\theta$ . However, the problem can be generalized for any set of DOAs  $\Theta$ , if the considered optimization problem is solved for the  $\theta \in \Theta$  with the maximum  $\cos^2 \theta$ , because for that particular  $\theta$ , the performance constraint is the tightest. The same generalizations can be applied for  $N$  and the SNR.

### B. Selection based on the CRB and the sidelobe level

Solving (7), we find a sparse sensor selection pattern that achieves some desired CRB. From the simulation results, it is seen that with an increasing threshold on the Fisher information (tighter lower bound on the variance), it starts selecting sensors from the boundary of the array. However, the beampattern of this type of arrays typically results in high sidelobes. To alleviate this problem we put another constraint

to control the spatial response in any direction  $\theta_{sl} \neq \theta$ . The additional constraint that should be added to the optimization problem in (7) is given by

$$|\mathbf{w}^T \tilde{\mathbf{a}}(\theta, \theta_{sl})| \leq \sqrt{\beta}, \quad (8)$$

where  $\tilde{\mathbf{a}}(\theta, \theta_{sl}) = [\tilde{a}_1(\theta, \theta_{sl}), \dots, \tilde{a}_M(\theta, \theta_{sl})]^T \in \mathbb{C}^{M \times 1}$  with entries  $\tilde{a}_i(\theta, \theta_{sl}) = e^{j \frac{2\pi}{\lambda} (p_i - p_c) (\sin \theta_{sl} - \sin \theta)}$ , and  $\beta$  is the desired level of sidelobe power in the specified angular region. The sidelobe constraint is related to the spatial response (in the direction of the sidelobes) of a matched filter beamformer for DOA  $\theta$  based only on the active sensors. The constraint can be applied for a specific set of angles  $\theta_{sl} \in \Theta_{sl}$ , outside the mainlobe area, where we would like to minimize the sidelobes.

The solution of (7) together with the constraint (8) for  $\theta_{sl} \in \Theta_{sl}$  gives the sparse sensor selection pattern which minimizes the sidelobe level in some specific angular region around the mainlobe along with achieving some desired CRB. The inevitable fact is that, when we increase  $\alpha$ , or decrease  $\beta$ , the designed array becomes less sparse.

### C. Sparsity-enhancing iterative algorithm

For a further improvement of the sparsity of  $\hat{\mathbf{w}}_1$ , the iterative re-weighted  $\ell_1$ -norm algorithm proposed in [12], [6] is adapted to suit our problem. In addition, during every iteration we refine  $\mu$  such that the relaxation of (7b) is almost equal to (6b).

The iterative algorithm goes as follows:

- 1) Initialize  $k = 0$ ,  $\|\hat{\mathbf{w}}^{(-1)}\|_0 = M (= \mu)$ , and  $\mathbf{u}^{(0)} = \mathbf{1}_M$ .
- 2) The following optimization problem is solved in the  $k$ -th iteration with the weighted objective function  $\mathbf{u}^{(k)T} \mathbf{w}^{(k)}$

$$(\hat{\mathbf{V}}^{(k)}, \hat{\mathbf{w}}^{(k)}) = \arg \min_{\mathbf{V}^{(k)} \in \mathbb{R}^{M \times M}, \mathbf{w}^{(k)} \in \mathbb{R}^M} \mathbf{u}^{(k)T} \mathbf{w}^{(k)} \quad (9a)$$

$$\text{s.t.} \quad \frac{\kappa'}{\|\hat{\mathbf{w}}^{(k-1)}\|_0} \cos^2(\theta) \mathbf{v}^T \mathbf{d} \geq \alpha, \quad (9b)$$

$$\begin{bmatrix} \mathbf{V}^{(k)} & \mathbf{w}^{(k)} \\ \mathbf{w}^{(k)T} & 1 \end{bmatrix} \succeq 0, \quad (9c)$$

$$[\mathbf{V}^{(k)}]_{ij} = [\mathbf{V}^{(k)}]_{ji} = v_{ij}, \\ i = 1, \dots, M-1, \quad j = i+1, \dots, M, \\ [\mathbf{V}^{(k)}]_{ii} = w_i^{(k)}, \quad i = 1, \dots, M, \quad (9d)$$

$$|\mathbf{w}^{(k)T} \tilde{\mathbf{a}}(\theta, \theta_{sl})| \leq \sqrt{\beta}, \quad \forall \theta_{sl} \in \Theta_{sl}. \quad (9e)$$

- 3) The elements of the weight vector  $u_i^{(k)}$  are updated as  $u_i^{(k)} = \frac{1}{\epsilon + |\hat{w}_i^{(k-1)}|}$ , for  $i = 1, \dots, M$ .
- 4) The iterations are stopped at  $k = k_{\max}$ , or on convergence, else increment  $k$  and go to step 2.

After the iterative algorithm, the selection vector  $\mathbf{w} \in \{0,1\}^M$  is generated by setting the nonzero values of  $\hat{\mathbf{w}}^{(k)}$  to 1. It is seen that, this operation does not affect the CRB constraint. The updating of the weights enhances the sparsity by forcing small entries of  $\hat{\mathbf{w}}^{(k)}$  to zero. The parameter  $\epsilon > 0$  which is used in the updating procedure is chosen to be very small to sparsify  $\hat{\mathbf{w}}^{(k)}$ . We use the MATLAB implementation of CVX [13], for solving the SDP problem (9).

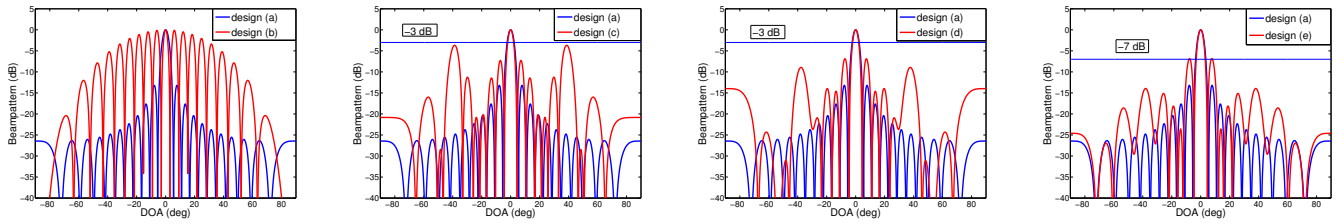


Fig. 1: Comparison of the beampatterns of the designed arrays with the ULA.

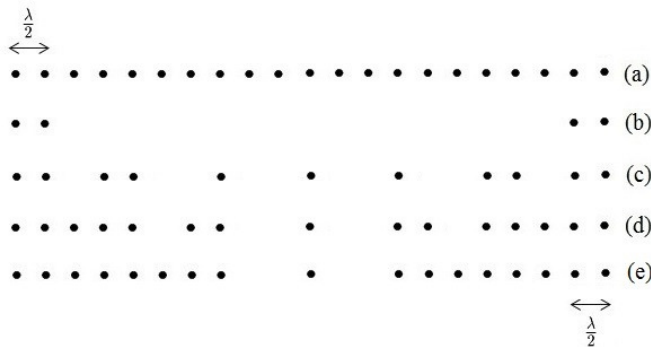


Fig. 2: ULA and non-uniform array designs.

#### IV. SIMULATION RESULTS

We consider a ULA of  $M = 21$  sensors placed with half-wavelength spacing. The aperture of the array  $A = p_M - p_1 = M\lambda/2$  is fixed. The parameters for the simulations are: SNR=10 dB,  $N = 1000$ . We normalize the inter-element spacing to  $\lambda/2 = 0.5$  and then set the coordinates of the sensors as  $p_i = i0.5$  m along a line. The aperture of the array of 21 sensors is then  $A = 10$ . The source signal is assumed to be impinging on the array from boreside, i.e.,  $\theta = 0^\circ$ . The parameters for the iterative algorithm are  $\epsilon = 10^{-8}$  and the number of iterations is  $k_{\max} = 20$ .

The generated arrays with the selected set of sensors are shown in Fig. 2. The array (a) is a ULA showing the available sensors. The design (b) is obtained by solving the iterative algorithm without any sidelobe constraints, with  $\alpha = 0.14F(\theta)$ , where the constant  $F(\theta)$  is the Fisher information of the full array in (3) calculated with the parameters mentioned above. Designs (c) and (d) are obtained by solving (9) with the same  $\beta = -3$  dB with  $\alpha = 0.14F(\theta)$  and  $\alpha = 0.21F(\theta)$ , respectively. The resulting array from solving the optimization problem (9) with  $\alpha = 0.14F(\theta)$  with  $\beta = -7$  dB is (e). The sidelobe region is kept fixed to  $\theta_{sl} \in [-90^\circ - 10^\circ] \cup [10^\circ 90^\circ]$  in designs (c), (d), and (e).

Finally, a comparison of the beampatterns of the resulting arrays with the full array are presented in Fig. 1. It is seen that design (a) has the best achievable CRB with minimal sidelobes. The design (b) is only CRB-optimal which reduces the number of sensors, but does not minimize the sidelobes. The beampatterns of designs (c), (d), and (e) show that with decreasing  $\beta$  or increasing  $\alpha$  (keeping any one of them fixed

and varying the other), more sensors are required to achieve the desired performance.

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