Parametrization of Hankel-norm approximants of time-varying systems

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The classical time-invariant Hankel-norm approximation problem is generalized to the time-varying context. The input-output operator of a time-varying bounded causal linear system acting in discrete time may be specified as a bounded upper-triangular operator T with block matrix entries T_{ij}. For such an operator T, we will define the Hankel norm as a generalization of the time-invariant Hankel norm. Subsequently, we describe all operators T' which are closer to T in (operator) norm than some prespecified error tolerance Γ, and whose upper triangular part admits a state realization of minimal dimensions. The upper triangular part of T' can be regarded as the input-output operator of a causal time-varying system that approximates T in Hankel norm.

1. INTRODUCTION

For time-invariant systems, the Hankel norm approximation problem (its minimal degree version) reads as follows [1]. Let $T(z) = t_0 + t_1 z + t_2 z^2 + \cdots$ be in the Hardy space H_∞, and define the Hankel operator $H_T = [t_{i+j+1}]_{i,j=0}^{\infty}$. Then, for a predefined error tolerance γ, find a transfer function $T_a(z)$ for which $\operatorname{rank} H_{T_a}$ is minimal, such that $\| H_{T-T_a} \| \leq \gamma$. Recall that the rank of H_T is the system order of T, *i.e.,* the minimal number of states that are required in a state realization of $T(z)$. A fundamental result, proven in [1], is that there exists an approximant T_a for which the state dimension is equal to the number of singular values of H_T which are larger than γ. The generalization to time-varying systems was derived by the authors in [2]. In this presentation, we will emphasize one of the results in this paper, namely the fact that all Hankel-norm approximants are described by a certain chain-fraction representation.

2. DEFINITIONS AND PRELIMINARY RESULTS

Define the space of *non-uniform* ℓ_2-sequences as follows. Let $M_i \in \mathbb{N} \setminus \{\infty\}$, for all integers i, and for each i define the vector space $\mathcal{M}_i = \mathbb{C}^{M_i}$. Then $\mathcal{M} = \cdots \times \mathcal{M}_i \times \cdots$ is a space of sequences whose entries are vectors of non-uniform dimensions, and

$$\ell_2^{\mathcal{M}} = \{ x \in \mathcal{M} : \| x \|_2 < \infty \}$$

[0] In U. Helmke e.a., editor, Systems and Networks: Mathematical Theory and Applications (Proc. Int. Symposium MTNS-93); volume 2, pp. 895-898, Regensburg, Germany, 1994. Akademie Verlag.

is the space of such sequences with bounded two-norm. Such sequences will represent signals in our theory. The space of bounded operators $T = [T_{ij}]_{i,j=-\infty}^{\infty}$ with entries T_{ij} which are $M_i \times N_j$ matrices acting on such sequences is

$$\mathcal{X}(\mathcal{M},\mathcal{N}) = [\ell_2^{\mathcal{M}} \to \ell_2^{\mathcal{N}}]\,.$$

We also define the space of upper operators as

$$\mathcal{U}(\mathcal{M},\mathcal{N}) \;=\; \{T \in \mathcal{X} : T_{ij} = 0, i < j\}$$

and likewise, the space $\mathcal{L}$ of lower and $\mathcal{D}$ of diagonal operators is defined. An operator $T \in \mathcal{X}(\mathcal{M},\mathcal{N})$ can be regarded as the input-output operator of a time-varying system acting on non-uniform sequences: an input sequence $u \in \ell_2^{\mathcal{M}}$ is mapped by T to an output sequence $y = uT \in \ell_2^{\mathcal{N}}$. The sequence $[T_{ij}]_{j=-\infty}^{\infty}$ (the i-th row of T) is the impulse response to an impulse at time i, and hence, for an LTI system, T has a Toeplitz structure. In the present notation, a causal system has an input-output operator $T \in \mathcal{U}$.

An operator $T \in \mathcal{U}$ has a time-varying state realization $\{A_k, B_k, C_k, D_k\}_{-\infty}^{\infty}$ if its block-entries are given by

$$T_{ij} \;=\; \begin{cases} 0\,, & i > j \\ D_i\,, & i = j \\ B_i A_{i+1} \cdots A_{j-1} C_j\,, & i < j \end{cases}$$

A realization is called strictly stable if $\lim_{n\to\infty} \sup_i \| A_{i+1} A_{i+1} \cdots A_{i+n} \|^{1/n} < 1$. In this case, the multiplication $y = uT$, with $u = [\cdots \;\; u_0 \;\; u_1 \;\; \cdots]$ and $y = [\cdots \;\; y_0 \;\; y_1 \;\; \cdots]$ is equivalent to the set of equations

$$\begin{array}{rcl} x_{k+1} & = & x_k A_k + u_k B_k \\ y_k & = & x_k C_k + u_k D_k \end{array} \qquad k = \cdots, 0, 1, \cdots,$$

in which x_k is introduced as the state. Note that state dimensions need not be constant.

In order to determine realizations with minimal state dimensions, we associate to an operator $T \in \mathcal{U}$ (or $T \in \mathcal{X}$) the collection of operators $\{H_k\}_{-\infty}^{\infty}$ which are submatrices of T:

$$H_k \;=\; [T_{k-i-1,k+j}]_{i,j=0}^{\infty} \;=\; \begin{bmatrix} T_{k-1,k} & T_{k-1,k+1} & \cdots \\ T_{k-2,k} & T_{k-2,k+1} & \\ \vdots & & \ddots \end{bmatrix}.$$

The H_k play the same role as the Hankel operator of T in the time-invariant case, although they do not possess a Hankel structure. In particular,

Theorem 1 ([3]) *Let $T \in \mathcal{U}$, $d_k := \operatorname{rank} H_k < \infty$ (all k). Then T admits a realization $\{A_k, B_k, C_k, D_k\}_{-\infty}^{\infty}$ where $A_k : d_k \times d_{k+1}$. This realization is minimal.*

In view of this theorem, we define $\operatorname{statedim}(T) := [\operatorname{rank} H_k]_{-\infty}^{\infty}$. We call T locally finite if all entries of this sequence are finite.

3. HANKEL NORM APPROXIMATION

The Hankel norm of $T \in \mathcal{X}$ is defined as

$$\| T \|_H \ := \ \sup_k \| H_k \| \,.$$

The Hankel norm is a seminorm, and weaker than the operator norm, as submatrices of a matrix have smaller norm than the matrix itself.

The time-varying Hankel-norm approximation problem can be formulated as follows. Given $T \in \mathcal{U}$ and a diagonal parameter operator $\Gamma \in \mathcal{D}$ ($\Gamma > 0$ and invertible), find $T' \in \mathcal{X}$ such that

$$\begin{array}{ll} (1) & \| \Gamma^{-1}(T - T') \| \ \le \ 1 \,, \\ (2) & \text{statedim}(T') \text{ is minimal (pointwise)}\,. \end{array}$$

Then T_a := (upper part of T') can be called a Hankel-norm approximant of T of minimal state dimension, as $\| \Gamma^{-1}(T - T_a) \|_H = \| \Gamma^{-1}(T - T') \|_H \le \| \Gamma^{-1}(T - T') \| \le 1$.

Theorem 2 ([2]) *Let $T \in \mathcal{U}$ be locally finite and have a strictly stable realization. Partition the singular values of $(H_{\Gamma^{-1}T})_k$ as $(\sigma_+)_{i,k} \le 1$, $(\sigma_-)_{i,k} > 1$, and suppose that* $\sup_{i,k} (\sigma_+)_{i,k} < 1$, $\inf_{i,k} (\sigma_-)_{i,k} > 1$. *Let N_k be the number of elements of the set $\{(\sigma_-)_{i,k}\}_i$. Then there exists an operator $T' \in \mathcal{X}$ satisfying*

$$\begin{array}{ll} (1) & \| \Gamma^{-1}(T - T') \| \ \le \ 1 \,, \\ (2) & \text{statedim}(T') \ \le \ [N_k]_{-\infty}^{\infty} \,. \end{array}$$

It is possible to show that $\text{statedim}(T')_k < N_k$ cannot occur. A suitable T' can be constructed by the following recipe [2]:

1. Determine an inner system $U \in \mathcal{U}$ (satisfying $UU^* = I$, $U^*U = I$) such that $UT^* \in \mathcal{U}$.
2. *Interpolation*: construct a J-unitary operator $\Theta \in \mathcal{U}$ (satisfying $\Theta^* J_1 \Theta = J_2$, $\Theta J_2 \Theta^* = J_1$ for certain signature operators $J_{1,2} \in \mathcal{D}$) such that

$$[U^* \ \ -T^*\Gamma^{-1}]\,\Theta \ =: \ [A' \ \ -B'] \in [\mathcal{U} \ \ \mathcal{U}]\,.$$

3. Define $T' = \Gamma\Theta_{22}^{-*}B'^{*} \ = \ T - \Gamma(\Theta_{12}\Theta_{22}^{-1})^* U$.

To outline the proof that this T' satisfies the two conditions in the theorem, let us remark that under the posed conditions on $\Gamma^{-1}T$ one can construct the operators U and Θ. In addition, one can show that $\| \Theta_{12}\Theta_{22}^{-1} \| < 1$ so that $\| \Gamma^{-1}(T - T') \| \ \le \ 1$. Finally, it is not hard to see from $T' = \Gamma\Theta_{22}^{-*}B'^{*}$ with $\Theta_{22}^{-*} \in \mathcal{X}$ and $B'^{*} \in \mathcal{L}$ that $\text{statedim}(T') \le \text{statedim}(\Theta_{22}^{-*})$. With more effort, one shows that there exists a Θ for which $\text{statedim}(\Theta_{22}^{-*})_k = N_k$, so that also the second requirement of the theorem is fulfilled.

U and Θ can be computed using state space techniques, and in this way a state realization of T_a can be obtained [2]. A suitable Θ can also be computed by a recursive generalized Schur procedure [4].

4. ALL APPROXIMANTS

The next issue is to determine *all* $T' \in \mathcal{X}$ satisfying the two conditions in theorem 2. The solution will be that all such T' are given by $T' = T + \Gamma S^* U$, where S is given by a linear fractional transformation of Θ and a free parameter S_L, which is upper and contractive (the previous solution is obtained by setting $S_L = 0$). In particular, the following two theorems hold true, showing that more, resp. all approximants are obtained.

Theorem 3 ([2]) *Let $T \in \mathcal{U}$, $\Gamma \in \mathcal{D}$ be as in theorem 2 and define U, Θ as before, where* $\text{statedim}(\Theta_{22}^{-*})_k = N_k$. *Let $S_L \in \mathcal{U}$, $\| S_L \| \le 1$. Put $S = (\Theta_{11} S_L - \Theta_{12})(\Theta_{22} - \Theta_{21} S_L)^{-1}$.*

Then $T' := T + \Gamma S^* U$ satisfies

$$\begin{aligned} &(1) \quad \| \Gamma^{-1}(T - T') \| \;\le\; 1\,,\\ &(2) \quad \text{statedim}(T') \;=\; [N_k]_{-\infty}^{\infty}\,. \end{aligned}$$

Theorem 4 ([2]) *Let T, Γ, U, Θ be as in theorem 3. Let $T' \in \mathcal{X}$ be any operator satisfying*

$$\begin{aligned} &(1) \quad \| \Gamma^{-1}(T - T') \| \;\le\; 1\,,\\ &(2) \quad \text{statedim}(T') \;\le\; [N_k]_{-\infty}^{\infty}\,. \end{aligned}$$

Define $S = U(T'^* - T^*)\Gamma^{-1}$ and $S_L = (\Theta_{11} S + \Theta_{12})(\Theta_{21} S + \Theta_{22})^{-1}$. Then

$$\begin{aligned} &S_L \in \mathcal{U}, \| S_L \| \;\le\; 1\,,\\ &S \;=\; (\Theta_{11} S_L - \Theta_{12})(\Theta_{22} - \Theta_{21} S_L)^{-1}\,. \end{aligned}$$

In fact, $\text{statedim}(T') = [N_k]_{-\infty}^{\infty}$, so that there are no approximants of order less than $[N_k]_{-\infty}^{\infty}$.

In this paper, we will only provide an outline of the proofs. It is straightforward to show that, in both theorems, $\| S_L \| \le 1 \;\Leftrightarrow\; \| S \| \le 1 \;\Leftrightarrow\; \Gamma^{-1}(T - T') \| \le 1$. The main point to prove in the first theorem is that T' has state dimensions as specified and in the second theorem that $S_L \in \mathcal{U}$. These proofs are related; the line of reasoning is as in [5], although the winding number argument is to be replaced by the following proposition:

Proposition 1 ([2]) *Let $A \in \mathcal{U}$, $A^{-1} \in \mathcal{X}$; $X \in \mathcal{X}$, $\| X \| < 1$.*
Let $N_k = \text{statedim(lower part of } A^{-1})_k^*$. *Then*

$$\begin{aligned} &\text{statedim(lower part of } (I - X)^{-1} A^{-1})_k^* \;=\; N_k + p_k\\ &\textit{iff } \text{statedim(lower part of } A(I - X))_k^* \;=\; p_k\,. \end{aligned}$$

The application of this proposition to theorem 3 is as follows. Put $A = \Theta_{22}$, $X = \Theta_{22}^{-1}\Theta_{21} S_L$, for any $S_L \in \mathcal{U}$, $\| S_L \| \le 1$. Then $(I - X)^{-1} A^{-1} = (\Theta_{22} - \Theta_{21} S_L)^{-1}$. Hence

$$\begin{aligned} &\text{statedim(lower part of } \Theta_{22}^{-1})_k^* = N_k \text{ and } \Theta_{22} - \Theta_{21} S_L \in \mathcal{U}\\ &\Rightarrow \quad \text{statedim(lower part of } (\Theta_{22} - \Theta_{21} S_L)^{-1})_k^* = N_k\,. \end{aligned}$$

This implies that $T'^* \Gamma^{-1} = (A' S_L + B')(\Theta_{22} - \Theta_{21} S_L)^{-1}$ has statedim(lower part of $T'^* \Gamma^{-1})_k^* \le N_k$. A similar argument gives equality.

REFERENCES

[1] V.M. Adamjan, D.Z. Arov, and M.G. Krein, "Analytic properties of Schmidt pairs for a Hankel operator and the generalized Schur-Takagi problem," *Math. USSR Sbornik*, vol. 15, no. 1, pp. 31–73, 1971. (transl. of *Iz. Akad. Nauk Armjan. SSR Ser. Mat. 6 (1971))*.

[2] P.M. Dewilde and A.J. van der Veen, "On the Hankel-norm approximation of upper-triangular operators and matrices," *Integral Equations and Operator Theory*, vol. 17, no. 1, pp. 1–45, 1993.

[3] A.J. van der Veen and P.M. Dewilde, "Time-varying system theory for computational networks," in *Algorithms and Parallel VLSI Architectures, II* (P. Quinton and Y. Robert, eds.), pp. 103–127, Elsevier, 1991.

[4] A.J. van der Veen and P.M. Dewilde, "On low-complexity approximation of matrices," *subm. Linear Algebra and its Applications*, 1992.

[5] J.A. Ball, I. Gohberg, and L. Rodman, *Interpolation of Rational Matrix Functions*, vol. 45 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, 1990.