WIDEBAND DIRECTION OF ARRIVAL ESTIMATION WITH SPARSE LINEAR ARRAYS

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ABSTRACT

This paper concerns wideband direction of arrival (DoA) estimation with sparse linear arrays (SLAs). We rely on the assumption that the power spectrum of the wideband sources is the same up to a scaling factor, which could in theory allow us to resolve not only more sources than the number of antennas but also more sources than the number of degrees of freedom (DoF) of the difference co-array of the SLA. We resort to the Jacobi-Anger approximation to transform the co-array response matrices of all frequency bins into a single virtual uniform linear array (ULA) response matrix. Based on the obtained model, two super-resolution DoA estimation approaches based on atomic norm minimization (ANM) are proposed, one with and one without prior knowledge of the power spectrum. Simulation results show that our proposed methods outperform the state of the art and are indeed capable of resolving more sources than the number of DoF of the difference co-array.

Index Terms— Wideband direction of arrival (DoA) estimation, sparse linear array (SLA), Jacobi-Anger approximation, atomic norm minimization (ANM)

I. INTRODUCTION

Wideband direction of arrival (DoA) estimation using a sensor array is an extensively studied technology and supports a wide range of applications in, for instance, wireless communication, acoustics and passive sonar [1]. Conventional methods rely on a bank of narrowband filters, which decompose the received wideband signal into several narrowband signals, and then employ subspace-based signal processing algorithms to obtain the DoA estimates. The simplest subspace-based wideband method is the incoherent signal subspace method (ISSM) [2], which applies narrowband techniques, such as MUSIC [3] and ESPRIT [4], independently to the outputs of the filter bank. Alternatively, in the coherent signal subspace method (CSSM) [5], focusing matrices are designed to combine the information from different frequency subbands, leading to improved performance compared to ISSM. The focusing schemes for CSSM are further developed in [6]. Inspired by compressed sensing (CS) theory [7], in the past decade some wideband DoA estimation approaches based on sparse signal recovery techniques have also been developed [8]–[10].

Most of the wideband DoA estimation methods have been confined to the case of uniform linear arrays (ULAs) and resolve up to $N-1$ sources with an N-element array. However, the topic of DoA estimation with more sources than sensors has been receiving considerable attention [11]–[13]. An efficient way to achieve this goal is to use a sparse linear array (SLA) and to construct a new difference co-array with more degrees of freedom (DoF) than that directly obtained from the physical SLA. From the co-array perspective, the minimum redundancy array (MRA) [14] and sparse ruler array (SRA) [15] have been considered as optimal SLA designs, yet their antenna locations cannot be computed in closed form. In the past decade, several more tractable SLA configurations have been proposed, such as the nested array [12] and the coprime array [13]. Based on these SLA configurations, most of the DoA estimation works focus on developing algorithms under the narrowband assumption [16]–[18]. For the wideband scenario, the DoA estimation problem for SLAs becomes more involved. In [19], a simple combined spatial smoothing MUSIC (SS-MUSIC) spectrum is constructed to exploit the spectral information from all frequency bins. In [20], a focusing Khatri-Rao (FKR) subspace-based approach is proposed, where the way to calculate the focusing matrices is similar to that in CSSM, but now extended to the difference co-array. Some DoA estimation methods based on grid-based CS and sparse reconstruction are also proposed for wideband sources [21]–[23], yet they suffer from leakage effects when the sources are off the grid. Note that for all the above methods, the number of sources to be recovered should be less than the number of DoF of the difference co-array.

In this paper, we focus on wideband DoA estimation with SLAs. In contrast to existing methods, we rely on the assumption that the power spectrum of the sources is the same up to a scaling factor, which is the case for many practical scenarios. Similar with previous works [19]–[23], the difference co-array response matrices for all frequency bins are constructed first. But as opposed to focusing, we resort to the Jacobi-Anger approximation from the manifold separation technique (MST) [24], [25] in array processing, to transform the difference co-array response matrices from the different frequency bins into a single virtual ULA response matrix. This transformation allows us to combine the data from different frequencies easily, and could in theory also resolve more sources than the number of DoF of the difference co-array. Based on the obtained model, we propose two super-resolution off-the-grid DoA estimation approaches based on atomic norm minimization (ANM) [26], one with and one without prior knowledge of the power spectrum. Simulation results show that, through efficiently merging the information from different frequency subbands, our proposed methods outperform the state of the art and are capable of resolving more sources than the number of DoF of the difference co-array.

II. SIGNAL MODEL

Consider K far-field, independent, and wideband sources impinging on an SLA with N antenna elements. For wideband processing, at each antenna, the signal is first sampled at Nyquist rate and partitioned into segments, and then a filter bank or an M-point sliding discrete Fourier transform (DFT) is applied to each segment to compute M frequency subbands. The noiseless received signal for the nth antenna and mth frequency can be written as

$$x_{n,m}[l] = \sum_{k=1}^{K} a_{n,m}(\theta_k) s_{k,m}[l],\ n \in \mathcal{N},\ m \in \mathcal{M}, \quad (1)$$

where $\mathcal{N} \triangleq \{1, \ldots, N\}$, $\mathcal{M} \triangleq \{1, \ldots, M\}$, $l \in \mathbb{N}^+$ denotes the index of the segment, $s_{k,m}[l]$ represents the source signal related to the kth source and mth frequency, $\theta_k \in [0, \pi)$ is the DoA of the kth source signal, and $a_{n,m}(\theta)$ represents the channel response at angle θ for the nth antenna and mth frequency. The channel response can generally be expressed as

$$a_{n,m}(\theta) \triangleq e^{-j2\pi d_n \cos(\theta)/\lambda_m},\ n \in \mathcal{N},\ m \in \mathcal{M}, \quad (2)$$

where d_n is the distance from the nth antenna to the first antenna, which is, for simplicity, an integer multiple of the basic element spacing d, and λ_m is the wavelength corresponding to the mth frequency f_m. For the nth antenna and mth frequency, the K source signals $\{s_{k,m}[l]\}_{k=1}^{K}$ are assumed to be mutually uncorrelated.

Stacking the signals of all antennas for the mth frequency, i.e, introducing $\boldsymbol{x}_m[l] \triangleq [x_{1,m}[l]\ x_{2,m}[l]\ \ldots\ x_{N,m}[l]]^T$ and $\boldsymbol{a}_m(\theta) \triangleq [a_{1,m}(\theta)\ a_{2,m}(\theta)\ \ldots\ a_{N,m}(\theta)]^T$, we obtain

$$\boldsymbol{x}_m[l] = \sum_{k=1}^{K} \boldsymbol{a}_m(\theta_k) s_{k,m}[l] = \boldsymbol{A}_m \boldsymbol{s}_m[l], \quad (3)$$

where $\boldsymbol{s}_m[l] \triangleq [s_{1,m}[l]\ s_{2,m}[l]\ \ldots\ s_{K,m}[l]]^T$ and $\boldsymbol{A}_m \triangleq [\boldsymbol{a}_m(\theta_1)\ \boldsymbol{a}_m(\theta_2)\ \ldots\ \boldsymbol{a}_m(\theta_K)]$. Computing the output covariance matrix $\boldsymbol{R}_m \triangleq E\{\boldsymbol{x}_m[l]\boldsymbol{x}_m^H[l]\}$, we obtain

$$\boldsymbol{R}_m = \boldsymbol{A}_m \mathrm{diag}\{\boldsymbol{\gamma}_m\} \boldsymbol{A}_m^H, \quad (4)$$

where $E\{\cdot\}$ represents the expectation operator, and $\boldsymbol{\gamma}_m = [\gamma_{1,m}\ \gamma_{2,m}\ \ldots\ \gamma_{K,m}]^T$ with $\gamma_{k,m}$ the source power related to the kth source and mth frequency, i.e., $\gamma_{k,m} \triangleq E\{|s_{k,m}[l]|^2\}$. Vectorizing this expression, we obtain

$$\tilde{\boldsymbol{r}}_m \triangleq \mathrm{vec}\{\boldsymbol{R}_m\} = \widetilde{\boldsymbol{B}}_m \boldsymbol{\gamma}_m, \quad (5)$$

where $\widetilde{\boldsymbol{B}}_m \triangleq (\boldsymbol{A}_m^* \circ \boldsymbol{A}_m)$ denotes the new array response matrix from the co-array perspective, with $(\cdot)^*$ and $\circ$ standing for the complex conjugate and Khatri-Rao product, respectively. Define the location set of the difference co-array as $\mathcal{D} \triangleq \{d_i - d_j\ :\ i, j \in \mathcal{N}\}$, and the cardinality of $\mathcal{D}$ as N_{co}, which indicates the DoF of the co-array, with $N^2 \geq N_{\mathrm{co}} \geq 2N - 1$. Denote $\{\xi_n\}_{n=1}^{N_{\mathrm{co}}}$ as the elements of $\mathcal{D}$ and let $\mathcal{N}_{\mathrm{co}} \triangleq \{1, \ldots, N_{\mathrm{co}}\}$. After removing the repeated rows in $\widetilde{\boldsymbol{B}}_m$ and the related entries in $\tilde{\boldsymbol{r}}_m$, we have

$$\boldsymbol{r}_m \triangleq \boldsymbol{J}\tilde{\boldsymbol{r}}_m = \boldsymbol{J}\widetilde{\boldsymbol{B}}_m \boldsymbol{\gamma}_m = \boldsymbol{B}_m \boldsymbol{\gamma}_m, \quad (6)$$

where $\boldsymbol{J} \in \{0,1\}^{N_{\mathrm{co}} \times N^2}$ is the corresponding selection matrix, and $\boldsymbol{B}_m \triangleq \boldsymbol{J}\widetilde{\boldsymbol{B}}_m$ is the (non-redundant) co-array response matrix related to the mth frequency. The kth column of $\boldsymbol{B}_m$ can be expressed as $\boldsymbol{b}_m(\theta_k) = [b_{1,m}(\theta_k)\ b_{2,m}(\theta_k)\ \ldots\ b_{N_{\mathrm{co}},m}(\theta_k)]^T$, with

$$b_{n,m}(\theta) \triangleq e^{-j2\pi\xi_n \cos(\theta)/\lambda_m},\ n \in \mathcal{N}_{\mathrm{co}},\ m \in \mathcal{M}. \quad (7)$$

The problem in this paper is to recover the continuous-valued DoAs $\{\theta_k\}_{k=1}^{K}$, given $\{\boldsymbol{r}_m\}_{m=1}^{M}$. Throughout the remainder of the paper, the following assumption will be adopted.

- A1: The power spectrum of all the K sources is the same up to a scaling, i.e., $\gamma_{k,m} = \alpha_k p_m$, with α_k the power of the kth source and p_m the normalized power spectrum of every source.

This is for instance the case in a wireless communication system where all sources use the same modulation format and pulse shaping functions [27]. Defining $\boldsymbol{\alpha} = [\alpha_1\ \alpha_2\ \ldots\ \alpha_K]^T$, we thus obtain $\boldsymbol{\gamma}_m = p_m\boldsymbol{\alpha}$. We will later on develop methods for the case $\{p_m\}_{m=1}^{M}$ is known as well as for the case it is unknown.

III. JOINT DATA MODEL

In order to merge the data models in (6) for all M different frequencies, we make the following assumption.

- A2: We assume that all co-array response matrices $\boldsymbol{B}_m$ can be transformed into a single virtual ULA response matrix $\boldsymbol{V} = [\boldsymbol{v}(\theta_1)\ \boldsymbol{v}(\theta_2))\ \ldots\ \boldsymbol{v}(\theta_K)]$, i.e., $\boldsymbol{B}_m = \boldsymbol{G}_m\boldsymbol{V}$, where $\boldsymbol{v}(\theta) \triangleq [e^{-j\theta N_{\mathrm{virt}}}\ \ldots\ e^{j\theta N_{\mathrm{virt}}}]^T$, $\boldsymbol{G}_m \in \mathbb{C}^{N_{\mathrm{co}} \times (2N_{\mathrm{virt}}+1)}$ denotes the corresponding transformation matrix which depends on the mth frequency f_m only, and N_{virt} is an odd number denoting the number of antennas in the virtual ULA.

One way to achieve this is by focusing, which generally requires an initial estimate of the DoAs or the signal subspace. We however rely on the more accurate Jacobi-Anger expansion, which provides a general infinite series expansion of exponentials of trigonometric functions in the basis of their harmonics [24], [25], and hence leads to a signal-independent transformation. Specifically, from the definition in (7), the (n,k)th entry of the co-array response matrix $\boldsymbol{B}_m$ can be written as

$$b_{n,m}(\theta_k) = \sum_{n_v=-\infty}^{\infty} j^{n_v} J_{n_v}\left(2\pi\frac{\xi_n}{\lambda_m}\right) e^{j\theta_k n_v} \quad (8)$$

where $J_{n_v}(\cdot)$ is the Bessel function of the first kind of order n_v. Note that although (8) indicates an infinite sum, the amplitude of $J_{n_v}(\cdot)$ decays very rapidly as the value of n_v increases beyond the argument of the related Bessel function

$J_{n_v}(\cdot)$. In practice, the infinite series can hence be truncated by considering only a limited number of modes as

$$b_{n,m}(\theta_k) \approx \sum_{n_v=-N_{\rm virt}}^{N_{\rm virt}} j^{n_v} J_{n_v}\left(2\pi \frac{\xi_n}{\lambda_m}\right) e^{j\theta_k n_v} = \boldsymbol{g}_{n,m}^T \boldsymbol{v}(\theta_k), \quad (9)$$

where $\boldsymbol{g}_{n,m} \triangleq [g_{-N_{\rm virt}}^{(n,m)} \ \ldots \ g_{N_{\rm virt}}^{(n,m)}]^T$, and $g_{n_v}^{(n,m)} \triangleq j^{n_v} \cdot J_{n_v}(2\pi\xi_n/\lambda_m)$. We can make the resulting truncation error arbitrarily small by increasing the number of modes. However, beyond some precision, increasing the number of modes does not increase the number of DoF of the virtual ULA and only increases the complexity. This specific precision is hard to determine and is usually chosen by some rule of thumb. For example, in [25], the lower bound on the mode order $N_{\rm virt}$ is determined as

$$N_{\rm virt} \geq \frac{2\pi}{\min_m\{\lambda_m\}} \cdot \max_n\{\xi_n\}. \quad (10)$$

Now, according to (9) and (10), the kth column of the co-array response matrix $\boldsymbol{B}_m$ can be expressed as $\boldsymbol{b}_m(\theta_k) = \boldsymbol{G}_m \boldsymbol{v}(\theta_k)$, where $\boldsymbol{G}_m \triangleq [\boldsymbol{g}_{1,m} \ \boldsymbol{g}_{2,m} \ \ldots \ \boldsymbol{g}_{N_{\rm co},m}]^T$, which corroborates Assumption A2.

Under Assumptions A1 and A2, we can rewrite (6) as

$$\boldsymbol{r}_m = p_m \boldsymbol{G}_m \boldsymbol{V}\boldsymbol{\alpha} = \boldsymbol{H}_m \boldsymbol{V}\boldsymbol{\alpha}, \quad (11)$$

where $\boldsymbol{H}_m \triangleq p_m \boldsymbol{G}_m$. Hence, up to the matrix factor $\boldsymbol{G}_m$ or $\boldsymbol{H}_m$, all frequencies yield the same model, which will be very useful for DoA estimation. On the other hand, it is still challenging to jointly utilize $\boldsymbol{r}_m$ from all frequencies in an efficient manner, because $\boldsymbol{G}_m$ (or $\boldsymbol{H}_m$) of size $N_{\rm co} \times (2N_{\rm virt}+1)$ is not invertible for large $N_{\rm virt}$. To tackle this challenge, a couple of effective wideband DoA estimators will be discussed in the next section.

IV. SUPER-RESOLUTION DOA ESTIMATION

Based on the joint data model derived in Section III, we here investigate super-resolution techniques for wideband DoA estimation. In Assumption A1, we have assumed that all K sources share the same normalized power spectrum $\{p_m\}_{m=1}^M$. In the following, both the scenarios with and without prior knowledge of the power spectrum $\{p_m\}_{m=1}^M$, are considered.

A. Known Power Spectrum

If the power spectrum $\{p_m\}_{m=1}^M$ is known *a priori*, we can stack the different vectors $\boldsymbol{r}_m$ into $\boldsymbol{r} \triangleq [\boldsymbol{r}_1^T \ \boldsymbol{r}_2^T \ \ldots \ \boldsymbol{r}_M^T]^T$ and merge the different equations (11) into $\boldsymbol{r} = \boldsymbol{H}\boldsymbol{V}\boldsymbol{\alpha}$, where $\boldsymbol{H} \triangleq [\boldsymbol{H}_1^T \ \boldsymbol{H}_2^T \ \ldots \ \boldsymbol{H}_M^T]^T$. We can then develop an algorithm to recover $\boldsymbol{\alpha}$ under the ANM framework. Note that the term $\boldsymbol{c} \triangleq \boldsymbol{V}\boldsymbol{\alpha}$ includes a linear combination of K complex sinusoids, and hence it has a sparse representation over the atom set $\mathcal{A} \triangleq \{\boldsymbol{v}(\theta) : \ \theta \in [0,\pi)\}$. Evidently, $\boldsymbol{c}$ has a sparse linear representation over $\mathcal{A}$. As a penalty function specially catered to the structure of the atom set $\mathcal{A}$, the atomic norm of $\boldsymbol{c}$ over $\mathcal{A}$ is defined as $\|\boldsymbol{c}\|_{\mathcal{A}} \triangleq \inf\{t > 0 : \ \boldsymbol{c} \in t \cdot \mathrm{conv}(\mathcal{A})\}$, where $\mathrm{conv}(\mathcal{A})$ denotes the convex hull of $\mathcal{A}$. We can first recover $\boldsymbol{c}$ by solving the ANM problem as

$$\min_{\boldsymbol{c}} \quad \frac{\lambda_1}{2}\|\boldsymbol{r} - \boldsymbol{H}\boldsymbol{c}\|_2^2 + \|\boldsymbol{c}\|_{\mathcal{A}}, \quad (12)$$

where λ_1 denotes the regularization parameter to balance the tradeoff between the ANM and the data fitting error. The joint ANM problem (12) can be represented in an equivalent semi-definite programming (SDP) form as

$$\begin{aligned} \min_{t,\boldsymbol{u},\boldsymbol{c}} \quad & \frac{\lambda_1}{2}\|\boldsymbol{r} - \boldsymbol{H}\boldsymbol{c}\|_2^2 + \mathrm{trace}(\mathcal{T}(\boldsymbol{u})) + t \\ s.t. \quad & \begin{bmatrix} t & \boldsymbol{c}^H \\ \boldsymbol{c} & \mathcal{T}(\boldsymbol{u}) \end{bmatrix} \succeq 0, \end{aligned} \quad (13)$$

where $\mathcal{T}(\boldsymbol{u})$ is a Hermitian Toeplitz matrix with the first column being $\boldsymbol{u}$. This SDP problem can be solved by some off-the-shelf solvers such as SeDuMi and SDPT3 [28], or some first-order fast algorithms such as the accelerated proximal gradient or alternating direction method of multipliers. Given $\boldsymbol{c}$ or $\mathcal{T}(\boldsymbol{u})$, DoA estimation can be performed using, for instance, any subspace-based method. In the simulations, root MUSIC [29] is used to obtain the final DoA estimates.

B. Unknown Power Spectrum

If the power spectrum $\{p_m\}_{m=1}^M$ is unknown, we can first transform (11) into $\boldsymbol{r}_m q_m = \boldsymbol{G}_m \boldsymbol{V}\boldsymbol{\alpha}$, where $q_m \triangleq 1/p_m$. Note that we here assume that only frequencies are considered for which p_m is not too small. Next, stacking the vectors $\boldsymbol{r}_m$ as $\boldsymbol{R} = \mathrm{blkdiag}\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_M\}$ and introducing $\boldsymbol{q} = [q_1 \ q_2, \ \ldots \ q_M]^T$, we can form $\boldsymbol{R}\boldsymbol{q} = \boldsymbol{G}\boldsymbol{V}\boldsymbol{\alpha}$, where $\boldsymbol{G} \triangleq [\boldsymbol{G}_1^T \ \boldsymbol{G}_2^T \ \ldots \ \boldsymbol{G}_M^T]^T$, and $\mathrm{blkdiag}\{\cdot\}$ denotes the block diagonal operator.

We now seek to jointly estimate the angles $\{\theta_k\}_{k=1}^K$ and the unknown power profile $\{p_m\}_{m=1}^M$. The unknown vectors $\boldsymbol{c}$ and $\boldsymbol{q}$ can be solved from the following least squares problem with atomic-norm regularization:

$$\min_{\boldsymbol{c},\boldsymbol{q}} \quad \frac{\lambda_2}{2}\|\boldsymbol{R}\boldsymbol{q} - \boldsymbol{G}\boldsymbol{c}\|_2^2 + \|\boldsymbol{c}\|_{\mathcal{A}}, \quad s.t. \ \ \mathbf{1}^T\boldsymbol{q} = 1, \quad (14)$$

where the linear constraint $\mathbf{1}^T\boldsymbol{q}$ is added to avoid the trivial solution $\boldsymbol{q} = \mathbf{0}$, with $\mathbf{1}$ and $\mathbf{0}$ denoting the vector of ones and zeros respectively. As before, the joint ANM problem (14) can be reformulated as an SDP problem as:

$$\begin{aligned} \min_{t,\boldsymbol{u},\boldsymbol{c},\boldsymbol{q}} \quad & \frac{\lambda_2}{2}\|\boldsymbol{R}\boldsymbol{q} - \boldsymbol{G}\boldsymbol{c}\|_2^2 + \mathrm{trace}(\mathcal{T}(\boldsymbol{u})) + t \\ s.t. \quad & \begin{bmatrix} t & \boldsymbol{c}^H \\ \boldsymbol{c} & \mathcal{T}(\boldsymbol{u}) \end{bmatrix} \succeq 0, \quad \mathbf{1}^T\boldsymbol{q} = 1. \end{aligned} \quad (15)$$

Based on $\boldsymbol{q}$ and $\boldsymbol{c}$, the power spectrum $\{p_m\}_{m=1}^M$ and the angles $\{\theta_k\}_{k=1}^K$ can be readily obtained. For the simulations, we use element-wise inversion to obtain the power spectrum and root MUSIC [29] to obtain the angles.

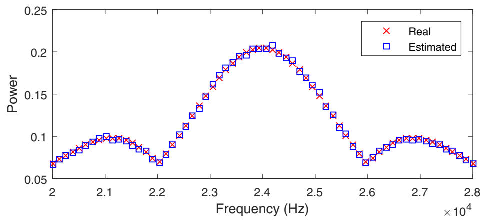

Fig. 1. Real and estimated power spectrum of each source, $L = 5000$ and SNR = 20 dB.

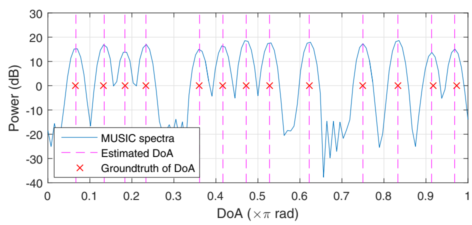

Fig. 2. MUSIC spectra, estimated DoAs and the groundtruth DoAs, $L = 5000$ and SNR = 20 dB.

V. Simulation Results

For the simulations, we will generate the received signals as a superposition of several harmonics corrupted by additive white Gaussian noise under a finite observation time. More specifically, we consider model (3) with the extra noise term $\boldsymbol{w}_m[l] \sim \mathcal{N}(\mathbf{0}, \sigma_w^2 \boldsymbol{I})$ and with sources distributed as $\boldsymbol{s}_m[l] \sim \mathcal{N}(\mathbf{0}, \text{diag}\{p_m \boldsymbol{\alpha}\})$. In total, L segements are considered. The signal-to-noise ratio (SNR) is defined as $\text{SNR} \triangleq \sum_{m=1}^{M} \left(\sum_{k=1}^{K} \gamma_{k,m} \right) / \left(M\sigma_w^2 \right)$. An unbiased estimate for the covariance matrix $\boldsymbol{R}_m$ can be obtained as $\hat{\boldsymbol{R}}_m = \left(\sum_{l=1}^{L} \boldsymbol{x}_m[l] \boldsymbol{x}_m^H[l] \right) / L$. Throughout the simulations, we consider a 2-level nested array of 4 antennas with locations $\{d_1, d_2, d_3, d_4\} = \{0, d, 2d, 5d\}$ where we set $d = 0.04$ m as the basic element spacing. Note that under this setup, the co-array will consist of $N_{\text{co}} = 11$ antennas. In addition, the speed of the signal wave is assumed to be $c = 340$ m/s and the wavelength corresponding to the mth frequency f_m is given by $\lambda_m = c/f_m$. Here we set $M = 64$ and consider evenly-spaced frequency points $\{f_m\}_{m=1}^{M}$ over the frequency range of the signal. In our experiments, for a fair comparison, we assume that the knowledge of K is available to all algorithms.

In the first simulation, each signal has the common center frequency $f_c = 24$ kHz and a common bandwidth of 8 kHz. The mode order of the Jacobi-Anger approximation is chosen as $N_{\text{virt}} = 120$, which satisfies (10). We consider the case where $K = 13$ uncorrelated wideband sources impinge on the 2-level nested array mentioned above. The DoAs of these 13 sources are given by $\{12, 24, 33, 42, 65, 75, 85, 95, 112, 135, 150, 165, 175\}\pi/180$. The normalized power spectrum of each source, i.e., $\{p_m\}_{m=1}^{M}$, is shown in Fig. 1 (marked with '×'), and, as stated in Assumption A1, we assume that the power spectrum of all K sources is the same up to a scaling. Here, we generate each scaling factor α_k independently with $\alpha_k \sim \chi^2(1)$. The number of snapshots is set to $L = 5000$, with SNR = 20 dB. We use the formulation proposed in Section IV-B with $\lambda_2 = 0.5$ to jointly recover the vector $\boldsymbol{c}$, which contains the information of the angles $\{\theta_k\}_{k=1}^{K}$, and the unknown power profile $\{p_m\}_{m=1}^{M}$. The estimated power spectrum is presented in Fig. 1 (marked with '□'), and Fig. 2 depicts the MUSIC spectrum of $\boldsymbol{c}$, the estimated DoAs and the groundtruth DoAs. We can see that, under this setup, the basic element spacing is larger than half the largest wavelength, and the number of DoF of the co-array is smaller than the number of sources, i.e., $d > \max_m\{\lambda_m/2\}$ and $N_{\text{co}} < K$. Still, our proposed DoA and power spectrum estimation method works well.

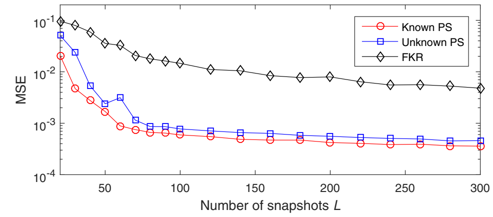

Fig. 3. MSE vs. the number of snapshots, SNR = 20 dB.

To better evaluate the performance, we calculate the mean square error (MSE) for the DoAs as $\text{MSE} \triangleq \sum_{k=1}^{K} |\theta_k - \hat{\theta}_k|^2$. In this example, we set the number of sources to $K = 5$ with DoAs given by $\{55, 65, 73, 105, 150\}\pi/180$. The frequency range of the sources is from 2 kHz to 8 kHz and the mode order of the Jacobi-Anger approximation is set as $N_{\text{virt}} = 40$. We here consider both the scenarios with and without prior knowledge of the power spectrum, and for our proposed methods in (13) and (15), the regularization parameters are set as $\lambda_1 = 0.1$ and $\lambda_2 = 0.5$, respectively. The FKR subspace approach [20] is introduced as a benchmark, where the true DoAs are used in the simulations to construct the focusing matrices. Fig. 3 depicts the MSE vs. the number of snapshots L, where we set SNR = 20 dB and the results are averaged over 1000 independent runs. Our proposed methods are referred to as the Known PS for (13) and Unknown PS for (15), respectively. We observe that our proposed methods achieve a higher DoA estimation accuracy than the FKR. Furthermore, as expected, Known PS provides better DoA estimates than Unknown PS but the gap is very small.

VI. Conclusions

In this paper we studied wideband DoA estimation with SLAs, exploiting the additional assumption that the power spectrum of the sources is the same up to a scaling factor. To combine the data from different frequencies, we resorted to the Jacobi-Anger approximation to transform the difference co-array response matrices form all frequency bins into a single virtual ULA response matrix. Simulation results showed that our proposed algorithms outperform the benchmark and are capable of resolving more sources than the number of DoF of the difference co-array.

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