DOA Estimation in Heteroscedastic Noise with sparse Bayesian Learning

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***Abstract*—We consider direction of arrival (DOA) estimation from long-term observations in a noisy environment. In such an environment the noise source might evolve, causing the stationary models to fail. Therefore a heteroscedastic Gaussian noise model is introduced where the variance can vary across observations and sensors. The source amplitudes are assumed independent zero-mean complex Gaussian distributed with unknown variances (i.e., source powers), leading to stochastic maximum likelihood (ML) DOA estimation. The DOAs are estimated from multi-snapshot array data using sparse Bayesian learning (SBL) where the noise is estimated across both sensors and snapshots.**

***Index Terms*—Heteroscedastic noise, sparse reconstruction**

I. Introduction

With long observation times, parameters of weak signals can be estimated in a noisy environment. Most analytic treatments analyze these cases assuming Gaussian noise with constant variance. For long observation times the noise process is likely to change with time leading to an evolving noise variance. This is called a heteroscedastic Gaussian process. While the noise variance is a nuisance parameter, it still needs to be estimated or included in the processing in order to obtain an accurate estimate of the parameters of the weak signals.

We resolve closely spaced weak sources when the noise power is varying in space and time. Specifically, we derive noise variance estimates and demonstrate this for compressive beamforming [1]–[3] using multiple measurement vectors (MMV or multiple snapshots). We solve the MMV problem using sparse Bayesian learning (SBL) [2], [4], [5]. Further details is in the paper [6] and demonstrated on real data [7].

We base our development on our fast SBL method [4], [5] which simultaneously estimates noise variances as well as source powers. For the heteroscedastic noise considered here, there could potentially be as many unknown variances as the number of observations. We estimate the unknown variances using approximate stochastic ML [8], [9] modified to obtain noise estimates even for a single observation.

Let $\mathbf{X} = [\mathbf{x}_1, \dots, \mathbf{x}_L] \in \mathbb{C}^{M\times L}$ be the complex source amplitudes, $x_{ml} = [\mathbf{X}]_{m,l} = [\mathbf{x}_l]_m$ with $m \in \{1, \cdots, M\}$ and $l \in \{1, \cdots, L\}$, at M DOAs (e.g., $\theta_m = -90^\circ + \frac{m-1}{M}180^\circ$) and L snapshots for a frequency ω. We observe narrowband waves on N sensors for L snapshots $\mathbf{Y} = [\mathbf{y}_1, \dots, \mathbf{y}_L] \in \mathbb{C}^{N\times L}$. A linear regression model relates the array data $\mathbf{Y}$ to the source amplitudes $\mathbf{X}$ as:

$$\mathbf{Y} = \mathbf{AX} + \mathbf{N}. \tag{1}$$

The dictionary $\mathbf{A}=[\mathbf{a}_1,...,\mathbf{a}_M]\in\mathbb{C}^{N\times M}$ contains the array steering vectors for all hypothetical DOAs as columns, Further, $\mathbf{n}_l \in \mathbb{C}^N$ is additive zero-mean circularly symmetric complex Gaussian noise, which is generated from a *heteroscedastic* Gaussian process $\mathbf{n}_l \sim \mathcal{CN}(\mathbf{n}_l; \mathbf{0}, \mathbf{\Sigma}_{\mathbf{n}_l})$. We assume that the covariance matrix is diagonal and parameterized as:

$$\mathbf{\Sigma}_{\mathbf{n}_l} = \sum_{n=1}^{N} \sigma^2_{n,l}\mathbf{J}_n = \mathrm{diag}(\sigma^2_{1,l}, \dots, \sigma^2_{N,l}), \tag{2}$$

where $\mathbf{J}_n = \mathrm{diag}(\mathbf{e}_n) = \mathbf{e}_n\mathbf{e}_n^T$ with $\mathbf{e}_n$ the nth standard basis vector. Note that the covariance matrices $\mathbf{\Sigma}_{\mathbf{n}_l}$ are varying over the snapshot index $l = 1, \dots, L$. The set of all covariance matrices are $\mathbf{\Sigma}_\mathbf{N} = \{\mathbf{\Sigma}_{\mathbf{n}_1}, \dots, \mathbf{\Sigma}_{\mathbf{n}_L}\}$. We consider three cases for the a priori knowledge on the noise covariance model (2):

I: We assume wide-sense stationarity of the noise in space and time: $\sigma^2_{n,l} = \sigma^2 =$ const. The model is homoscedastic.

II: We assume wide-sense stationarity of the noise in space only, i.e., the noise variance for all sensor elements is equal across the array, $\sigma^2_{n,l} = \sigma^2_{0,l}$ and it varies over snapshots. The noise variance is heteroscedastic in time (across snapshots).

III: No additional constraints other than (2). The noise variance is heteroscedastic across both time and space (sensors and snapshots.)

We assume $M>N$ and thus (1) is underdetermined. In the presence of only few stationary sources, the source vector $\mathbf{x}_l$ is K-sparse with $K\ll M$. We define the lth active set $\mathcal{M}_l = \{m \in \mathbb{N} | x_{ml} \neq 0\}$, and assume $\mathcal{M}_l=\mathcal{M}=\{m_1,...,m_K\}$ is constant across all snapshots l. Also, we define $\mathbf{A}_\mathcal{M}\in\mathbb{C}^{N\times K}$ which contains only the K "active" columns of $\mathbf{A}$.

We assume that the complex source amplitudes x_{ml} are independent both across snapshots and across DOAs and follow a zero-mean circularly symmetric complex Gaussian distribution with DOA-dependent variance γ_m, $m = 1, \dots, M$,

$$p(x_{ml}; \gamma_m) = \begin{cases} \delta(x_{ml}), & \text{for } \gamma_m = 0 \\ \frac{1}{\pi\gamma_m}\mathrm{e}^{-|x_{ml}|^2/\gamma_m}, & \text{for } \gamma_m > 0 \end{cases}, \tag{3}$$

$$p(\mathbf{X}; \boldsymbol{\gamma}) = \prod_{l=1}^{L}\prod_{m=1}^{M} p(x_{ml}; \gamma_m) = \prod_{l=1}^{L} \mathcal{CN}(\mathbf{x}_l; \mathbf{0}, \mathbf{\Gamma}), \tag{4}$$

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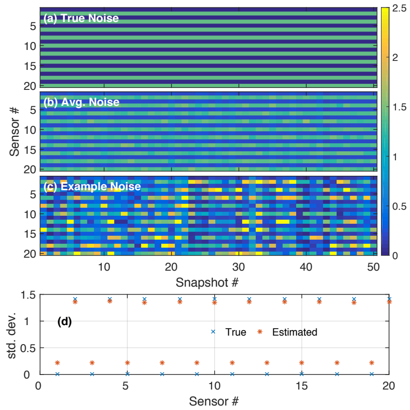

Fig. 1. Single source at DOA $-3°$, array SNR = 0 dB, noise standard deviation statistics: (a) true noise parameters, (b) average estimated noise parameters from SBL (100 simulations), (c) a typical SBL estimate, and (d) average across simulations and snapshots.

i.e., the source vector $\mathbf{x}_l$ at each snapshot $l\in\{1,\cdots,L\}$ is multivariate Gaussian with potentially singular covariance matrix,

$$\boldsymbol{\Gamma} = \text{diag}(\boldsymbol{\gamma}) = \mathsf{E}[\mathbf{x}_l\mathbf{x}_l^H;\boldsymbol{\gamma}], \tag{5}$$

as $\text{rank}(\boldsymbol{\Gamma})=\text{card}(\mathcal{M})=K\leq M$ (typically $K \ll M$). Note that the diagonal elements of $\boldsymbol{\Gamma}$, i.e., $\boldsymbol{\gamma}\geq\mathbf{0}$, represent source powers. When the variance $\gamma_m=0$, then $x_{ml}=0$ with probability 1. This likelihood function is identical to the Type II likelihood function (evidence) in standard SBL [2], [4] which is obtained by treating $\boldsymbol{\gamma}$ as a hyperparameter. The estimates $\hat{\boldsymbol{\gamma}}$ and $\hat{\boldsymbol{\Sigma}}_{\mathbf{N}}$ are obtained by maximizing the likelihood,

$$(\hat{\boldsymbol{\gamma}}\ \hat{\boldsymbol{\Sigma}}_{\mathbf{N}}) = \underset{\boldsymbol{\gamma}\geq 0,\ \boldsymbol{\Sigma}_{\mathbf{N}}}{\arg\max} \log p(\mathbf{Y};\boldsymbol{\gamma}\ \boldsymbol{\Sigma}_{\mathbf{N}}). \tag{6}$$

The goal is thus to solve (6) and the active DOAs $\mathcal{M}$ is where $\hat{\boldsymbol{\gamma}}$ 0. The SBL algorithm solves (6) by iterating between the source power estimates $\hat{\boldsymbol{\gamma}}$ derived in this section and the noise variance estimates $\hat{\boldsymbol{\Sigma}}_{\mathbf{N}}$. Assuming γ_m^{old} and $\boldsymbol{\Sigma}_{\mathbf{y}_l}$ given (from previous iterations) we obtain the following fixed point iteration for the γ_m [4] ($b = 0.5$):

$$\gamma_m^{\text{new}} = \gamma_m^{\text{old}} \left(\frac{\sum_{l=1}^{L} |\mathbf{y}_l^H \boldsymbol{\Sigma}_{\mathbf{y}_l}^{-1} \mathbf{a}_m|^2}{\sum_{l=1}^{L} \mathbf{a}_m^H \boldsymbol{\Sigma}_{\mathbf{y}_l}^{-1} \mathbf{a}_m} \right)^b. \tag{7}$$

II. Example

An example statistic of the heteroscedastic noise standard deviation is shown in Fig. 1 for a 20 element array with a single source. The standard deviation for each sensor is either 0 or $\sqrt{2}$ (Fig. 1a). The estimates of the standard deviation are in Figs. 1b, 1c). Average of estimated noise (Fig. 1b) resembles well the true noise (Fig. 1a) whereas the sample standard deviation estimate (Fig. 1c) has high variability—each estimate is based on just one observation. Given many simulations and snapshots, however, the mean of the estimated standard deviation is close to the true noise (Fig. 1d). Three noise cases are simulated: (a) Noise Case I: constant noise standard deviation over snapshots and sensors, (b) Noise Case II: standard deviation changes across snapshots with $\log_{10}\sigma_l \sim \mathcal{U}(-1{,}1)$, and (c) Noise Case III: standard deviation changes across both snapshots and sensors with $\log_{10}\sigma_{n,l} \sim \mathcal{U}(-1{,}1)$.

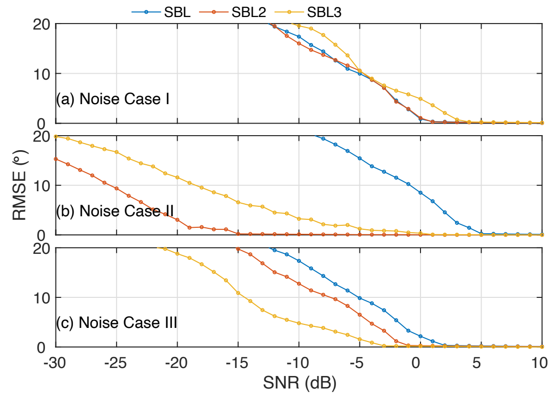

Fig. 2. RMSE vs. SNR with the three sources at $\{-3°, 2°, 50°\}$ and power $\{10, 22, 20\}$ dB.

In Fig. 2, we consider three sources located at $[-3, 2, 50]°$ with power $[10, 22, 20]$ dB. The complex source amplitude is stochastic and there is additive heteroscedastic Gaussian noise with SNR variation from -35 to 10 dB. The $N{=}20$ elements sensor array with half-wavelength spacing observe $L{=}50$ snapshots. The angle space grid $[-90 : 0.5 : 90]°$ ($M{=}360$). The single-snapshot array signal-to-noise ratio (SNR) is $\text{SNR}{=}10\log_{10}[\mathsf{E}\{\|\mathbf{A}\mathbf{x}_l\|_2^2\}/\mathsf{E}\{\|\mathbf{n}_l\|_2^2\}]$. The root mean squared error (RMSE) of the DOA estimates over 100 noise realizations is used for evaluating the algorithms.

The simulation shows that for Noise Case III (Fig. 2c) best results are obtained when estimating the full noise covariance matrix (green line, SBL3). Thus, the simulation demonstrates that estimating the noise carefully gives improved DOA estimation at low SNR.

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