BOUND ON THE ESTIMATION GRID SIZE FOR SPARSE RECONSTRUCTION IN DIRECTION OF ARRIVAL ESTIMATION

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ABSTRACT

A bound for sparse reconstruction involving both the signal-to-noise ratio (SNR) and the estimation grid size is presented. The bound is illustrated for the case of a uniform linear array (ULA). By reducing the number of possible sparse vectors present in the feasible set of a constrained ℓ_1-norm minimization problem, ambiguities in the reconstruction of a single source under noise can be reduced. This reduction is achieved by means of a proper selection of the estimation grid, which is naturally linked with the mutual coherence of the sensing matrix. Numerical simulations show the performance of sparse reconstruction with an estimation grid meeting the provided bound demonstrating the effectiveness of the proposed bound.

***Index Terms*—** direction of arrival (DOA) estimation, sensing matrix, uniform linear array, compressed sensing, sparse reconstruction

1. INTRODUCTION

The main goal of sparse-signal processing (SSP) is the reconstruction of the parameter vector $\mathbf{x} \in \mathbb{C}^{N\times 1}$, given that it is k-sparse, with $k \ll N$. One of the advantages of SSP is the possibility of perfect recovery of $\mathbf{x}$ from a reduced number of measurements $M < N$. This kind of processing is commonly known as compressed sensing (CS) [1], however SSP is not limited to the case of CS. In general the sparse reconstruction of $\mathbf{x}$ from a set of linear measurements

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}, \tag{1}$$

where $\mathbf{A}$ is the sensing matrix and $\mathbf{n}$ is the measurement noise, can be posed as a non linear optimization problem [1]

$$\min_{\mathbf{x}} \quad ||\mathbf{x}||_0 \quad \text{s.t.} \ ||\mathbf{A}\mathbf{x} - \mathbf{y}||_2^2 \leq \epsilon. \tag{2}$$

A common choice for ϵ is the noise power $\sigma_N^2 = ||\mathbf{n}||_2^2$ when a good estimate is known a priori. To solve (2) efficiently, the non-convex ℓ_0-norm is commonly relaxed to the convex ℓ_1-norm [2]

$$\min_{\mathbf{x}} \quad ||\mathbf{x}||_1 \quad \text{s.t.} \ ||\mathbf{A}\mathbf{x} - \mathbf{y}||_2^2 \leq \epsilon. \tag{3}$$

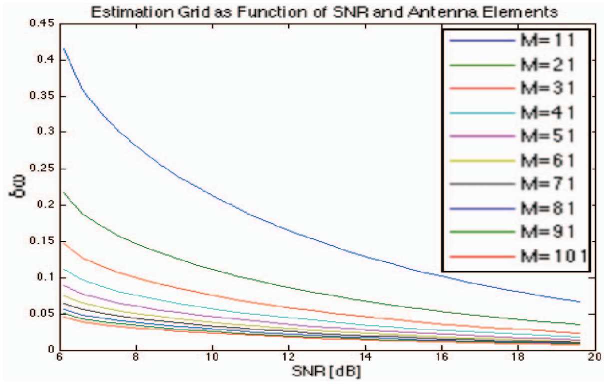

Fig. 1: Effect of the number of antenna elements M and SNR on δ^*.

When certain properties are met by the sensing matrix $\mathbf{A}$, the theory of CS can be applied to (3) [3]. Hence, the sensing matrix $\mathbf{A}$ has to be properly designed. For instance, in radar signal processing, where the model in (1) can be used to describe a sparse scene, i.e., small number of targets, $\mathbf{A}$ can be designed to suit CS and sparse reconstruction for different SNR values. However, in most of the cases, the columns of $\mathbf{A}$ are selected based on the fast Fourier transform (FFT), leading to a uniform estimation grid. As the SNR is crucial for the resolution [4][5], different scenes should lead to different grid size choices. This implies that given an SNR level, the system should be able to adjust its resolution accordingly. Several works exist on the fundamental theory of sparse reconstruction and CS [2][3][6]. However, the particularities of parameter estimation with respect to the available SNR of the system when sparse reconstruction is used are not fully discussed. As the estimation grid, defining the columns of $\mathbf{A}$, is related to both the resolution of the underlying signal processing and the quality of reconstruction [7], in this work we explore the precise relation between the SNR, fundamental for radar systems, with the estimation grid size defining a non-uniform resolution system. In this paper, an argument based on the idea of reducing the feasible set of the constrained ℓ_1-norm minimization problem is used to provide a bound which reduces the number of sparse vectors that could lead to ambiguities in the reconstruction. The bound is applied to DOA

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estimation for a single target under the radar signal model. The sparse reconstruction problem to be solved is discussed in Section 2, followed by the derivation of the bound on the optimal estimation grid based on the available SNR. In Section 3, preliminaries about the radar signal model are provided, particularly for the case of DOA estimation. Experimental results for DOA estimation using SSP are obtained by means of Monte Carlo simulations for both single-target and multiple-targets scenarios in Section 4. Conclusions and future work directions are discussed at the end of the paper.

2. BOUND ON ESTIMATION GRID

2.1. SSP-DOA Model and Sensing Matrix Coherence

Consider the particular case of DOA estimation over a uniform linear array. The data model is given by [8]

$$\mathbf{y} = \mathbf{A}_\theta \mathbf{x} + \mathbf{n} \tag{4}$$

where $\mathbf{x} \in \mathbb{C}^{N\times 1}$ contains the direction of arrival of the targets present in the scene, which is assumed to be k-sparse, i.e., there are k non-zero entries in $\mathbf{x}$. The sensing matrix $\mathbf{A}_\theta \in \mathbb{C}^{M\times N}$ is the array manifold matrix of the antenna array with M elements and an estimation grid with cardinality N. Every column in $\mathbf{A}_\theta$ can be seen as the array steering vector $\mathbf{a}(\theta)$ at a particular direction θ in the estimation grid $\boldsymbol{\theta} = [\theta_0, \ldots, \theta_{N-1}]^T$. For a ULA with half wavelength spacing, the normalized steering vector is given by

$$\mathbf{a}(\theta) = \frac{1}{\sqrt{M}}[1, e^{j\pi \sin(\theta)}, \ldots, e^{j\pi(M-1)\sin(\theta)}]^T. \tag{5}$$

The definition of the mutual coherence of the sensing matrix $\mathbf{A}_\theta$ in terms of its Gram matrix is given by [1]

$$\mu(\mathbf{A}_\theta) = \max_{i\neq j} |\mathbf{G}_{ij}| \tag{6}$$

where $\mathbf{G} = \mathbf{A}_\theta^H \mathbf{A}_\theta$ and $\mathbf{G}_{ij}$ is the (i,j)-th element of the matrix $\mathbf{G}$. For the case of a ULA, any inner product between the i-th and j-th columns of the sensing matrix is given by

$$\begin{aligned} \mathbf{a}^H(\theta_i)\mathbf{a}(\theta_j) &= \frac{1}{M}\sum_{k=0}^{M-1} e^{j\pi k(\sin(\theta)-\sin(\theta_i))} \\ &= \frac{1}{M}\sum_{k=0}^{M-1} e^{jk\delta_{ij}} \end{aligned} \tag{7}$$

here $\delta_{ij} \triangleq \pi(\sin(\theta_j) - \sin(\theta_i))$. Then, a closed form expression for the entries of the Gram matrix in this particular case can be found in terms of δ_{ij} as

$$|\mathbf{G}_{ij}| = \frac{1}{M}\left|\frac{\sin(0.5M\delta_{ij})}{\sin(0.5\delta_{ij})}\right|. \tag{8}$$

In the next part, (8) will be used to design the optimal estimation grid for a given SNR.

2.2. Minimum Estimation Grid Size

In order to guide the design of the estimation grid, the reconstruction process and its constraints should be considered. For that purpose, the problem posed in (3) will be used to establish an optimal grid design for different SNR conditions. The arguments presented here for defining the estimation grid size are based on the idea of reducing the cardinality of the feasible set. If this set is not tight enough, i.e., it contains vectors with low correlation, the optimization problem in (3) could accept solutions different from the true DOA vector leading to a degradation of the reconstruction. In this work, the case of targets on the estimation grid is only considered. In any other case, the bound derived here does not guarantee any reconstruction performance. Consider the true value for the DOA vector to be $\mathbf{x}_{\text{true}}$. Then the constraint in (3) is satisfied for any k-sparse vector $\mathbf{x}$ if and only if

$$\|\mathbf{A}\mathbf{x} - \mathbf{y}\| = \|\mathbf{A}\mathbf{x} - \mathbf{A}\mathbf{x}_{\text{true}} - \mathbf{n}\| = \|\mathbf{A}(\mathbf{x} - \mathbf{x}_{\text{true}}) - \mathbf{n}\| \leq \sqrt{\epsilon} \tag{9}$$

For the case of a single source, $k = 1$ and $\epsilon = \|\mathbf{n}\|^2$, by using the triangle inequality we obtain

$$\|\mathbf{a}_i \alpha_{\mathbf{x}} - \mathbf{a}_j \alpha_{\mathbf{x}_{\text{true}}}\| \leq 2\sqrt{\epsilon} \tag{10}$$

where $\mathbf{a}_i$ is the i-th column of $\mathbf{A}$ and $\alpha_{\mathbf{x}}$ is the signal amplitude of the source in $\mathbf{x}$. From the last expression, the optimal solution for the amplitude $\alpha_{\mathbf{x}}$ is given by

$$\alpha_{\mathbf{x}} = \alpha_{\mathbf{x}_{\text{true}}} \mathbf{a}_i^H \mathbf{a}_j \tag{11}$$

Substituting the previous expression, we can find solutions that are different from the true solution if and only if

$$\|\alpha_{\mathbf{x}_{\text{true}}}(\mathbf{a}_i(\mathbf{a}_i^H \mathbf{a}_j) - \mathbf{a}_j)\| \leq 2\sqrt{\epsilon} \tag{12}$$

For simplicity let us assume a unitary power source, $|\alpha_{\mathbf{x}_{\text{true}}}|^2 = 1$, which means that the SNR is equal to ϵ^{-1}. As a result the last expression can be written in terms of the elements of the matrix $\mathbf{G}$ as

$$|\mathbf{G}_{ij}| \geq \sqrt{1 - \frac{4}{SNR}}, \text{ for } SNR \geq 4, \tag{13}$$

and otherwise $|\mathbf{G}_{ij}| \geq 0$. The inequality in (13) provides a bound for the minimum coherence $\mathbf{G}_{ij}$ as a function of the SNR which leads to ambiguities in the feasible set. In other words, the bound from (13) implies that errors in reconstruction will occur when the sensing matrix has columns with a coherence equal to or higher than the bound as in that case the feasible set contains other vectors than the true solution. By inserting (8) in (13), which leads to

$$\frac{1}{M}\left|\frac{\sin(0.5M\delta_{ij})}{\sin(0.5\delta_{ij})}\right| \geq \sqrt{1 - \frac{4}{SNR}}, \quad \text{for } SNR \geq 4 \tag{14}$$

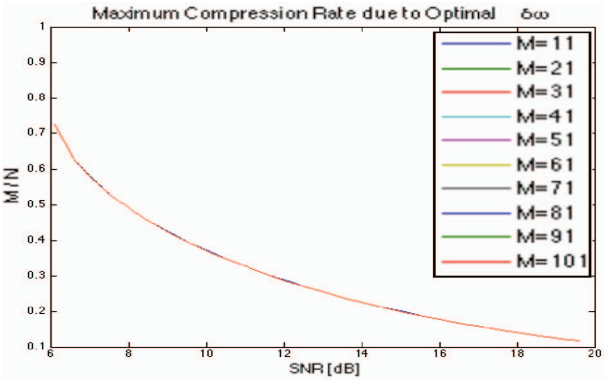


Fig. 2: Theoretical compression rate (M/N) of the sensing matrix $\mathbf{A}$ as function of M and SNR. $N = 2\pi/\delta^*$.

it is possible to observe that for a given SNR an optimal estimation grid can be found. For simplicity, let us assume an estimation grid that is uniform in $\omega = \pi \sin(\theta)$. In that case, the uniformity of the grid will lead (8) to only depend on the optimal estimation grid size δ^*. In order to obtain practical results, an approximation of (14) is made for small grid sizes, i.e., $\delta \approx 0$, and a sufficiently high SNR (SNR$\geq$ 6dB). A first order Taylor series expansion is used for the sinusoidal terms. The approximate optimal $\delta^{*(M)}_{SNR}$, for a given SNR and number of antenna elements M, is considered to be the largest value of δ which meets the following bound

$$\delta^3(\frac{0.5^3}{6}(\sqrt{1-\frac{4}{SNR}}-M^2)) + \delta(0.5(1-\sqrt{1-\frac{4}{SNR}})) \geq 0 \quad (15)$$

The inequality (13) reveals the super-resolution capabilities of the sparse reconstruction as $|\mathbf{G}_{ij}| \rightarrow 1$ for $SNR \rightarrow \infty$. This implies that only ambiguities in the feasible set occur when $\mathbf{A}$ contains highly correlated columns. The $\delta^{*(M)}_{SNR}$ is shown in Fig. 1 as a function of the SNR and the number of antenna elements M. The theoretical compression rates, ratio between the number of columns M and measurements N, as a function of SNR is shown in Fig. 2. By our choice for $\delta^{*(M)}_{SNR}$, it is shown in Fig. 2 that the compression rate is independent of M. Any sensing matrix $\mathbf{A}$ with an estimation grid size greater than δ^*_{SNR} ensures a feasible set containing a unique vector. Hence, there is no vector that leads to ambiguities in the reconstruction for a single target on the estimation grid.

3. EXTENSION TO RADAR SIGNAL PROCESSING

Consider the baseband signal $y(t)$ at a single antenna element, i.e., the total received backscattering from the scene after quadrature demodulation, to be given by [9]

$$y(t) = \int\int x(\tau,\omega)u(t-\tau)e^{-j\omega t}d\tau d\omega + n(t) \quad (16)$$

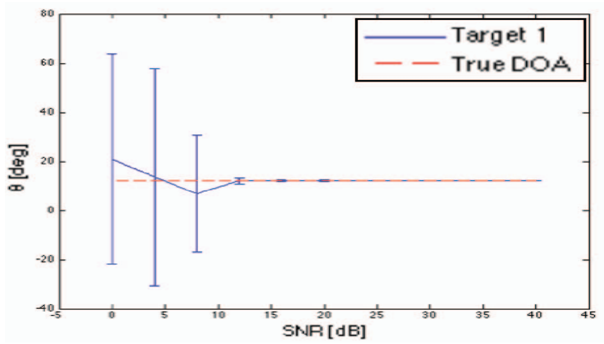


Fig. 3: SSP-DOA estimation for a source at $\theta = 12^o$ with $\delta = \delta^{*(11)}_{10dB} = 0.2$ for SNR ranging from 0 to 40dB and $M = 11$.

where the constant phase term has been included in the reflectivity term $x(\tau,\omega)$ and the noise is considered temporal and spatially white after sampling with $\mathbf{n} \sim \mathcal{CN}(0,\gamma\mathbf{I})$. By allowing a discretization leading to an estimation grid $[\tau_0,\tau_m] \times [\omega_0,\omega_l]$, where τ is the time delay and ω is the Doppler frequency, it is possible to express (16) (single channel) by means of a linear model as

$$\mathbf{y} = \mathbf{A}_{\tau,\omega}\mathbf{x} + \mathbf{n} \quad (17)$$

where the i-th column of $\mathbf{A}_{\tau,\omega}$ is a shifted version, in time and frequency, of the sent signal due to a target at a range and speed given by the pair (τ_i,ω_i). If the far field assumption is met, all the columns in $\mathbf{A}_{\tau,\omega}$ have the same ℓ_2-norm. The vector $\mathbf{x}$ contains the reflectivity of the targets present in the scene, which under realistic assumptions, is known to be sparse (small number of targets), Thus, SSP can be applied to (17) to reconstruct $x(\tau,\omega)$. Notice that this model can easily be extended to the multichannel case, i.e., signals impinging on an antenna array, by means of the Kronecker product ($\otimes$). For the angle-range-Doppler case the sensing matrix can be expressed as [9]

$$\mathbf{A}_{\tau,\omega,\boldsymbol{\theta}} = \mathbf{A}_{\boldsymbol{\theta}} \otimes \mathbf{A}_{\tau,\omega} \quad (18)$$

with $\mathbf{A}_{\boldsymbol{\theta}}$ being a matrix of stacked array response vectors from the grid $[\theta_0,\theta_k]$ where θ represents the azimuth angle observable from an ULA. In this work, simulations in an ULA for the azimuth angle only case will be the target of our study.

4. EXPERIMENTAL RESULTS

In this section the theoretical results for the optimal estimation grid size for the sensing matrix $\mathbf{A}$ are tested by means of Monte Carlo simulations. For this purpose, 100 instances of problem (3) are solved using YALL1 [10] for different SNR conditions. All results are reported within one sample standard deviation at both sides. In the first simulation, a signal is emitted by a source at direction $\theta = 12^o$ impinging on a uniform linear array (ULA) with $M = 11$ elements. Temporally and spatially white noise $\mathbf{n}$ is considered with variance ϵ. The SNR $= 1/\epsilon = 1/\|\mathbf{n}\|^2$ under test is from 0dB to 40dB.

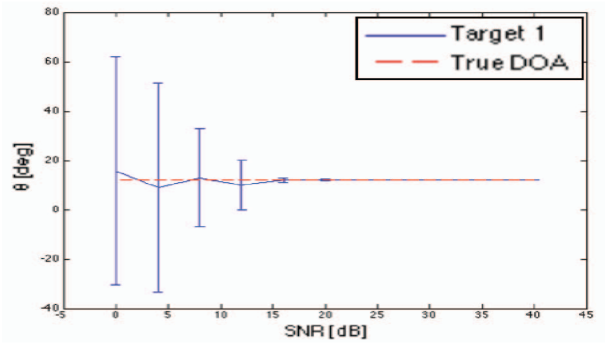
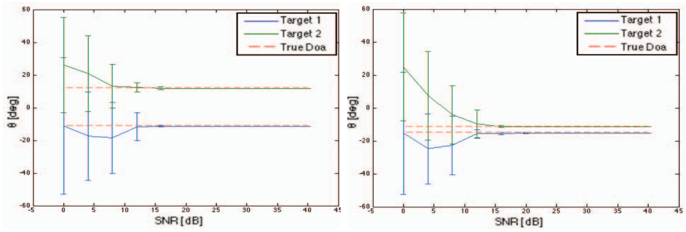


Fig. 4: SSP-DOA estimation for a source at $\theta = 12^o$ with $\delta = \delta^{*(11)}_{16dB} = 0.1$ for SNR ranging from 0 to 40dB and $M = 11$.

In order to solve (3), a sensing matrix is used with estimation grid size δ equal to the solution of (15) for $M = 11$ and an SNR of 10dB, i.e., $\delta = \delta^{*(11)}_{10\text{dB}} = 0.2$. The result in Fig. 3 illustrates the properties of the bound in (13) w.r.t. reconstruction errors. For $\delta = \delta^{*(11)}_{10dB}$ the SSP-DOA estimation becomes unbiased at an SNR higher than 10dB and its variance vanishes drastically, leading to perfect reconstruction as the SNR increases. In addition, a compression ratio $M/N \approx 0.33$ is attained. When $\delta = 0.1$ is used to build the sensing matrix $\mathbf{A}_\theta$ perfect reconstruction requires a higher SNR as shown in Fig. 4. In this case, the estimation grid offers its best performance when the SNR$\geq$ 16dB as suggested from the results of (15) illustrated in Fig. 1.To illustrate the reconstruction when more than one source is present, an experiment is carried out with two well-separated sources at $[-11^o, 12^o]$. The estimation grid size is chosen as $\delta = \delta^{*(11)}_{10dB}$ with an SNR ranging from 0dB to 40dB. Fig. 5(a) shows the reconstruction performance. Even though both sources are clearly identified for $SNR \geq$ 10dB, the reconstruction presents a higher variance than the single source case. However, perfect reconstruction is achieved as the SNR increases as expected. The same experiment is repeated with sources in contiguous estimation cells. Now the sources are found at directions $[-11^o, -15^o]$. The same ULA, δ and SNR range are used for the simulation. Fig. 5(b) shows that even when the sources are in contiguous estimation cells, SSP-DOA is able to approximately keep the guarantees given by (14).



(a) $\boldsymbol{\theta} = [-11^o, 12^o]^T$ (b) $\boldsymbol{\theta} = [-11^o, -15^o]^T$

Fig. 5: SSP-DOA estimation for two sources at $\boldsymbol{\theta}$ with $\delta = \delta^{*(11)}_{10dB} = 0.2$ for SNR ranging from 0 to 40dB and $M = 11$.

5. CONCLUSIONS

In this paper, a bound for sparse reconstruction describing the relation between the estimation grid size δ of a sensing matrix of a ULA for DOA estimation and the SNR is provided. By means of a constrained ℓ_1-norm minimization problem, an argument based on the reduction of the feasible set is used in order to reduce reconstruction errors. The number of sparse vector elements inside the n-ball which can cause reconstruction ambiguities are reduced by a proper selection of δ. Using a first order approximation for the sinusoidal terms of the closed-form for the mutual coherence of a ULA, an expression for computing the optimal grid size for given SNR conditions and number of antenna elements M is found. Simulation results have illustrated the performance of the SSP-DOA estimation when the optimal grid size is chosen to build $\mathbf{A}$ showing the practicality of the bound. Obtaining similar bounds for non-linear arrays, e.g., nested arrays, co-prime arrays, etc., and its relation with resolution limits is currently under investigation.

6. REFERENCES

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