DIRECTION OF ARRIVAL ESTIMATION BASED ON INFORMATION GEOMETRY

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ABSTRACT

In this paper, a new direction of arrival (DOA) estimation approach is devised using concepts from information geometry (IG). The proposed method uses geodesic distances in the statistical manifold of probability distributions parametrized by their covariance matrix to estimate the direction of arrival of several sources. In order to obtain a practical method, the DOA estimation is treated as a single-variable optimization problem, for which the DOA solutions are found by means of a line search. The relation between the proposed method and MVDR beamformer is elucidated. An evaluation of its performance is carried out by means of Monte Carlo simulations and it is shown that the proposed method provides improved resolution capabilities at low SNR with respect to MUSIC and MVDR.

Index Terms— direction of arrival (DOA) estimation, information geometry, uniform linear array, MUSIC, MVDR

1. INTRODUCTION

The problem of estimating the direction of arrival (DOA) of sources from the covariance matrix of received measurements is a well known problem [1]. Most of the current methods are based on subspace techniques or exploit characteristics of the structure present in the covariance matrix [2], [3] [4]. However, none of those methods considers the geometry present in the space of probability distributions parametrized by their covariance matrix. Recent work [5] has raised attention towards the usage of information geometry to describe the manifold in which probability distributions live and links with several fields have been established (e.g., neural networks [6], [7], optimization [8], [9]). It has been shown that when using this framework, robust estimation of covariance matrices is possible [5]. In addition, several applications and fundamental theory in radar systems [10], [11] and machine learning [12], [13] have been devised using concepts of information geometry. However, at the best of the knowledge of the authors, no work exists with respect to DOA estimation. The ideas of information geometry first introduced by Rao [14] and later formally developed by Cencov [15], constitute a framework that considers probability densities as structure of differential geometry. This approach allows to build a distance between two parametrized distributions that is invariant to non-singular transformation of the parameters [16]. As the distance is based on the Fisher information matrix (FIM), the results derived from information geometry are tightly linked with fundamental results in estimation theory, such as the celebrated Cramér-Rao lower bound (CRLB). Using as a base the geometry of multivariate Gaussian normal distributions (MGNDs), a method taking into account distances between probability distributions parametrized by hermitian positive definite (HPD) matrices (the coordinate system for this statistical manifold) is proposed for DOA estimation. The idea is the usage of the Riemannian metric proposed by IG, which is nothing more than the Fisher information matrix, to measure the *closeness* between different possible arrival angles. In this paper, we introduce a new DOA estimation method based on geodesic distances coming from the framework of IG. By measuring how close two distributions are, using these distances, our DOA approach becomes a linear search when the space of matrices to explore is constrained to be the set of rank-one matrices. The structure of the paper is given as follows. First, some preliminary information related to the basics of information geometry, particularly for the case of the MGND is presented in Section 2. In Section 3, the antenna model used throughout the work is introduced and the proposed method is presented. In Section 4, some light is shed on the relation between the MVDR beamformer and the proposed method. The DOA estimation method proposed in Section 3 is evaluated by means of Monte Carlo analysis in Section 5 and its results are compared with the ones from MUSIC and MVDR. Finally, Section 6 provides the conclusions of our work and possible future research directions.

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2. INFORMATION GEOMETRY OF COVARIANCE MATRICES

First consider an n-dimensional multivariate model for a set of measurements $\mathbf{x}$ given by

$$\mathbf{x} \sim \mathcal{CN}(\mathbf{0}, \mathbf{R}(\boldsymbol{\theta})) \tag{1}$$

where $\boldsymbol{\theta}$ is the parameter vector containing the unique elements of the matrix $\mathbf{R}(\boldsymbol{\theta})$. The likelihood function of the data given the unknown true covariance is given by

$$p(\mathbf{x}|\mathbf{R}(\boldsymbol{\theta})) = \frac{1}{\pi^n det(\mathbf{R}(\boldsymbol{\theta}))} e^{\{-tr(\mathbf{x}^H \mathbf{R}^{-1}(\boldsymbol{\theta})\mathbf{x})\}} \tag{2}$$

where $det(\cdot)$ and $tr(\cdot)$ denote the determinant and trace of a matrix. From now on, the notation expressing the dependency of $\mathbf{R}$ on $\boldsymbol{\theta}$ will be omitted when the relation is clear. For the multivariate normal distribution, the elements of the Fisher information matrix for the parameter vector $\boldsymbol{\theta}$ are given by [17]

$$G_{ij} = -E[\frac{\partial^2 \ln p(\mathbf{X}|\mathbf{R})}{\partial\theta_i\theta_j}] = tr[(\mathbf{R}^{-1}\partial_i\mathbf{R}) \cdot (\mathbf{R}^{-1}\partial_j\mathbf{R})] \tag{3}$$

where ∂_i is $\partial/\partial\theta_i$. Using the Fisher information matrix as metric tensor following the ideas of information geometry, the differential of a path length is given by [18]

$$d\mathbf{s}^2 = tr\{(\mathbf{R}^{-1}d\mathbf{R})^2\} = tr[(d\ln\mathbf{R})^2] \tag{4}$$

The generalization of straight lines in manifolds is given by geodesics, curves that achieve the shortest distance between two points $\mathbf{R}_1$ and

$\mathbf{R}_2$. The geodesic distance is then given by

$$d(\mathbf{R}_1, \mathbf{R}_2) = \min_{\gamma_{\mathbf{R}_1}^{\mathbf{R}_2}} \left\{ \int_{\gamma_{\mathbf{R}_1}^{\mathbf{R}_2}} d\mathbf{s} \right\} \tag{5}$$

where $\gamma_{\mathbf{R}_1}^{\mathbf{R}_2}$ is a path joining $\mathbf{R}_1$ and $\mathbf{R}_2$. By endowing the statistical manifold $\mathcal{M}$ for the space of probability distributions parametrized by their covariance matrix with the proposed metric, it is possible to treat $\mathcal{M}$ as a Riemannian manifold [18]. The introduced metric has some interesting properties:

1. Invariant to the group $GL(n)$ action

$$\mathbf{S}^H\mathbf{R}\mathbf{S},\ \mathbf{S} \in GL(n)$$

where $GL(n)$ is the set of $n \times n$ invertible matrices.

2. Invariant to matrix inversion

$$d(\mathbf{R}_1, \mathbf{R}_2) = d(\mathbf{R}_1^{-1}, \mathbf{R}_2^{-1})$$

Using the proposed metric, the geodesic distance between two probability distributions parametrized by the HPD matrices $\mathbf{R}_1$ and $\mathbf{R}_2$ is given by [18]

$$d(\mathbf{R}_1, \mathbf{R}_2) = \sum_{i=1}^{n} (\log a_i)^2 \tag{6}$$

where $a_1, ..., a_n$ are the roots of $det(\lambda\mathbf{R}_1 - \mathbf{R}_2)$.

As shown in [19] the distance between probability distributions expressed in (6) results in the natural Riemannian distance of the set of all n-by-n HPD matrices $\mathcal{P}_n$.

$$d(\mathbf{R}_1, \mathbf{R}_2) \triangleq d(p(\mathbf{x}|\mathbf{R}_1), p(\mathbf{x}|\mathbf{R}_2)) \tag{7}$$

By using the intrinsic distance of the statistical manifold, benefits in performance can be expected. As the cone of the symmetric matrices is not a vector space, using other distances, e.g., Euclidean distance, may lead to a degraded performance [19]. Motivated by this issue, the distance in (6) is used in order to estimate the angle of arrival of different sources from the measurements of an antenna array where the data follows a model similar to (1).

3. PROPOSED DOA ESTIMATION BASED ON INFORMATION GEOMETRY

3.1. Antenna Array Model

Assume there are D uncorrelated signals with equal power σ_s^2 and zero mean impinging from directions $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_D]^T$ on a uniform linear antenna array with M elements. The received signal vector of the antenna array at time k can then be expressed as

$$\mathbf{x}[k] = \sum_{i=1}^{D} \mathbf{a}(\theta_i) s_i[k] + \mathbf{n}[k] = \mathbf{A}\mathbf{s}[k] + \mathbf{n}[k] \tag{8}$$

where $\mathbf{A} = [\mathbf{a}(\theta_1) \ldots \mathbf{a}(\theta_D)] \in \mathbb{C}^{M\times D}$ is the array manifold matrix and $\mathbf{s}[k] = [s_1[k], \ldots, s_D[k]]^T$ represents the signal vector. The noise vector $\mathbf{n}[k] \sim \mathcal{CN}(\mathbf{0}, \sigma_n^2\mathbf{I}_M)$ is considered to be independent and identically distributed Gaussian noise. Furthermore, the i-th column of $\mathbf{A}$ contains the array vector response for the i-th source given by

$$\mathbf{a}(\theta_i) = [1, \psi_i, \ldots, \psi_i^{M-1}] \tag{9}$$

where

$$\psi_i = \exp(j2\pi\frac{l}{\lambda}\sin(\theta_i)) \tag{10}$$

with l being the distance between the antenna elements.

The true covariance matrix $\mathbf{R}_{xx} = E\{\mathbf{x}[k]\mathbf{x}^H[k]\}$ of the received data is given by

$$\begin{aligned}\mathbf{R}_{xx} &= \sigma_s^2 \sum_{i=1}^{d} \mathbf{a}(\theta_i)\mathbf{a}^H(\theta_i) + \sigma_n^2\mathbf{I}_M \\ &= \sigma_s^2\mathbf{A}\mathbf{A}^H + \sigma_n^2\mathbf{I}_M\end{aligned} \tag{11}$$

where $E\{\cdot\}$ denotes the mathematical expectation.

3.2. Algorithm Description

Under the previous model and the IG distance between probability distributions discussed in Section 2, the problem of DOA estimation can be stated equivalently as the following optimization problem

$$\begin{aligned}\min_{\tilde{\mathbf{R}},\tilde{\mathbf{A}}\in\mathcal{A}} \quad & d(\mathbf{R}_{xx}, \tilde{\mathbf{R}}) \\ s.t \quad & \tilde{\mathbf{R}} = \tilde{\mathbf{A}}\tilde{\mathbf{A}}^H\end{aligned} \tag{12}$$

Here $d(\cdot,\cdot)$ denotes the IG distance between probability distributions parametrized by their covariance matrices as given by (6). The problem above tries to find the probability distribution $p(\mathbf{x}|\tilde{\mathbf{R}})$ that is *closest* (in the information geometry sense) to the distribution described by the true covariance matrix, provided that $\tilde{\mathbf{R}} = \tilde{\mathbf{A}}\tilde{\mathbf{A}}^H$ and $\tilde{\mathbf{A}} \in \mathcal{A}$ where $\mathcal{A}$ is the set of feasible array manifold matrices given the array element positions and number of sources. Similar to the least squares (LS) approach, problem (12) minimizes an error measure. However, the Euclidean distance of LS is substituted by a more *natural* distance, the IG distance.

Assuming that the array response vector function $\mathbf{a}(\theta)$ is known, the feasible set $\mathcal{A}$ is the only thing that needs to be defined in order to solve (12). As the number of sources is usually unknown a priori, the set $\mathcal{A}$ cannot be easily defined. However, as the space of Hermitian positive semi-definite matrices is a convex cone whose interior contains the cone of the HPD matrices, (7) can be used as a projected distance towards the interior of the cone with respect to the rank-one components of the, possibly rank deficient ($M > D$), Gram matrix of the array manifold matrix

$$\mathbf{A}\mathbf{A}^H = \sum_{i=1}^{D} \mathbf{a}(\theta_i)\mathbf{a}^H(\theta_i) \tag{13}$$

Hence, a straightforward feasible set for the optimization problem can be designed as

$$\mathcal{A}_1 = \{\mathbf{a}(\phi),\ \phi \in [-\pi/2, \pi/2]\} \tag{14}$$

When the number of sources or the covariance properties are known, others feasible sets enforcing rank or structure in $\tilde{\mathbf{R}}$ can be used to solve (12), e.g., Toeplitz, circular, etc. In the rest of the paper, we will only discuss results for $\mathcal{A}_1$. By using the distance between the rank-one matrix $\tilde{\mathbf{R}}(\phi)$ for an angle ϕ, i.e.,

$$\tilde{\mathbf{R}}(\phi) = \mathbf{a}(\phi)\mathbf{a}^H(\phi) \tag{15}$$

and the full rank covariance matrix of the received data, the problem in (12) leads to a direction of arrival estimation algorithm consisting on a linear search for maximizers of

$$f(\phi) = \frac{1}{d(\hat{\mathbf{R}}_{xx}, \tilde{\mathbf{R}}(\phi))},\ \phi \in [-\pi/2, \pi/2] \tag{16}$$

where $d(\cdot,\cdot)$ denotes the IG distance computed from the generalized eigenvalue problem. Since the true covariance matrix $\mathbf{R}_{xx}$ is unavailable in practice, in (16) it has been replaced by the sample covariance matrix

$$\hat{\mathbf{R}}_{xx} = \frac{1}{K}\sum_{k=0}^{K-1}\mathbf{x}[k]\mathbf{x}^H[k] \tag{17}$$

where K denotes the number of data snapshots used to estimate the matrix.

Finally, as the set of generalized eigenvalues of $(\hat{\mathbf{R}}_{xx}, \tilde{\mathbf{R}}(\phi))$ is called the spectrum of the pencil $(\hat{\mathbf{R}}_{xx}, \tilde{\mathbf{R}}(\phi))$, the proposed DOA method is referred to as IGPencil in the rest of the paper.

4. EQUIVALENCE OF IGPENCIL AND MVDR

In this section, the relationship between the proposed method and minimum variance distortionless response (MVDR) beamforming is discussed. In Section 3, the IGPencil method was devised using a projected distance from the cone of positive definite matrices of size $M \times M$, $\mathcal{S}^M_{++}$, to the cone of positive semi-definite matrices of rank one and size $M \times M$, $\mathcal{S}^M_{+,1}$. The proposed one-dimensional search function through the set of angles ϕ was given in (16). In the particular case of $\mathcal{A}_1$, we can leverage the rank properties of $\tilde{\mathbf{R}}(\phi)$ to reduce the distance expression to a more straightforward one. As the a_i parameter are the solution to the generalized eigenvalue problem, it is possible to define them through the relation

$$\tilde{\mathbf{R}}(\phi)\mathbf{v}_i = \hat{\mathbf{R}}_{xx}a_i\mathbf{v}_i \tag{18}$$

where a_i are the generalized eigenvalues and $\mathbf{v}_i$ are the corresponding generalized eigenvectors. Assuming $\hat{\mathbf{R}}_{xx}$ is non-singular, in case of enough temporal snapshots, (18) can be rewritten as

$$\hat{\mathbf{R}}^{-1}_{xx}\tilde{\mathbf{R}}(\phi)\mathbf{v}_i = a_i\mathbf{v}_i \tag{19}$$

which poses a standard eigenvalue problem. From the property

$$\text{rank}(\hat{\mathbf{R}}^{-1}_{xx}\tilde{\mathbf{R}}(\phi)) = \text{rank}(\tilde{\mathbf{R}}(\phi)) = 1 \tag{20}$$

it clear that (19) only has one eigenvalue distinct from zero. By using the trace property

$$tr(\mathbf{R}) = \sum_{i=1}^{M}\lambda_i \tag{21}$$

where λ_i are the corresponding eigenvalues of the square matrix $\mathbf{R}$, the only non-zero eigenvalue used for the IG distance is given by

$$a = tr(\hat{\mathbf{R}}^{-1}_{xx}\tilde{\mathbf{R}}(\phi)) \tag{22}$$

which can be rearranged by the invariance of the trace under cyclic permutation as

$$a = \mathbf{a}(\phi)^H\hat{\mathbf{R}}^{-1}_{xx}\mathbf{a}(\phi) \tag{23}$$

By rewriting (16) in terms of (23) and (7) we obtain

$$f(\phi) = \frac{1}{(\log \mathbf{a}(\phi)^H\hat{\mathbf{R}}^{-1}_{xx}\mathbf{a}(\phi))^2},\ \phi \in [-\pi/2, \pi/2] \tag{24}$$

As both $log(x)$ and x^2 are jointly monotonically increasing functions comparable results are expected $\forall x \geq 1$ from the expression

$$f(\phi) = \frac{1}{\mathbf{a}^H(\phi)\hat{\mathbf{R}}^{-1}_{xx}\mathbf{a}(\phi)},\ \phi \in [-\pi/2, \pi/2] \tag{25}$$

However, when the denominator of (25) falls below one, i.e., $\mathbf{a}^H(\phi)\hat{\mathbf{R}}^{-1}_{xx}\mathbf{a}(\phi) \lesssim 1$, probably by interference between close sources or a high level of noise, due to the nature of the non-linear transformation $log(\cdot)^2$ different results between MVDR and IGPencil are expected. In Section 5 benefits in resolution when (24) is used for detecting close sources are shown.

5. EXPERIMENTAL RESULTS

In this section, some numerical results are shown and comparisons to MUSIC and MVDR are illustrated. In order to provide a fair comparison, the proposed algorithm is tested using Monte Carlo simulations under different SNR conditions. For all the simulations a uniform linear array (ULA) of $M = 11$ elements is used. In the first simulation, we generate a set of $M - 1 = 10$ sources with equal unitary power and a uniform separation in degrees within the range $\Omega = [-\pi/3, \pi/3]$. Temporally and spatially white noise is considered. In addition, a set of $K = 100$ time snapshots are collected and an SNR of 10dB is assumed. The results from MUSIC, MVDR and IGPencil are shown in Fig. 1. Observe how the performance of the proposed method is comparable to MUSIC and MVDR. In addition, all the degrees of freedom available in traditional MUSIC and MVDR are also present in IGPencil as all the $M - 1 = 10$ signals are detected by the method. Next, we evaluate the overall performance of the proposed direction of arrival method through a set of 1000 Monte Carlo experiments where two sources are buried in noise under different SNR conditions. The sources are located at directions $\boldsymbol{\theta} = [-20^o, 30^o]^T$. The SNR range under test is from -20dB to 20dB. As before, the same ULA with half-wavelength spacing and $M = 11$ elements is used. The statistical performance of the three methods is presented in Fig. 2. Observe how IG Pencil, which presents an identical performance as MVDR at high SNR, has a higher total mean square error (MSE) for well-resolvable sources at low SNR. These results agree with the equivalence discussed in Section 4. When the experiment is repeated with closer sources $\boldsymbol{\theta} = [-20^o, -23^o]^T$, Fig. 3 and Fig. 4 show how IGPencil outperforms both MUSIC and MVDR in terms of resolution. As the SNR increases, IGPencil tends to the MVDR performance which leads to a degraded performance until the crit-

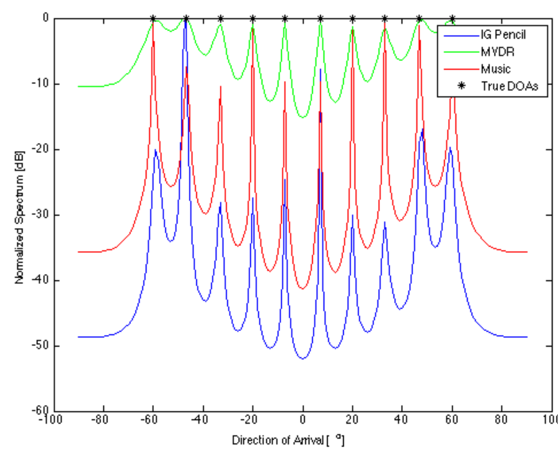

Fig. 1. Comparison between IGPencil, MUSIC and MVDR spectrum for 10 sources, 11 antenna elements and an SNR of 10dB

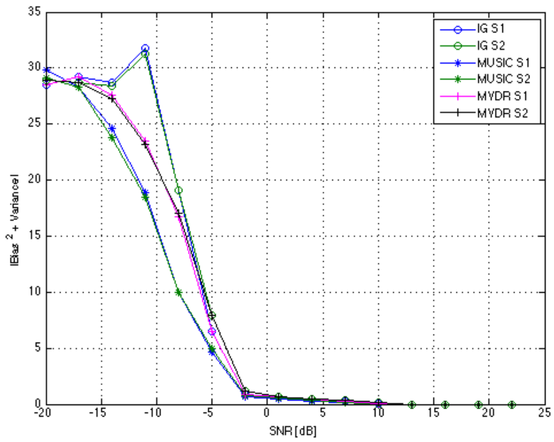

Fig. 2. Statistical performance for two sources at $[-20^o, 30^o]$ by MUSIC, IGPencil and MVDR for SNR ranging from -20 to 20dB

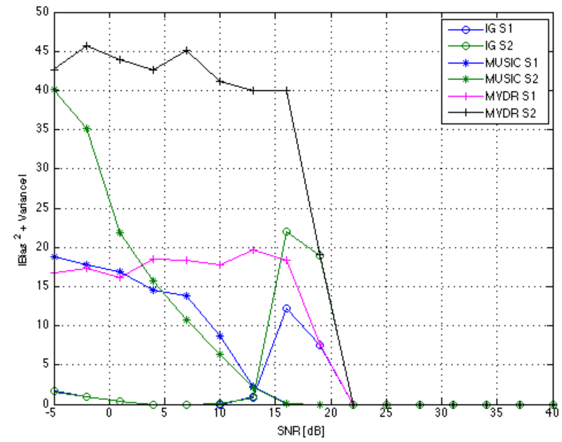

Fig. 4. Statistical performance for two sources at $[-20^o, -23^o]$ by MUSIC, IGPencil and MVDR for SNR ranging from -5 to 40dB

ical SNR for resolving both sources is achieved. By means of the gain in resolution and the IG distance, more sources than the available degrees of freedom can be detected. In Fig. 5, 13 sources at $\boldsymbol{\theta} = [-60^o, -50^o, -34^o, -31^o, -20^o, -5^o, -8^o, 5^o, 10^o, 25^o, 41^o, 44^o, 60^o]^T$ with an SNR of 10dB are detected by IGPencil, where neither MUSIC or MVDR are able to do so.

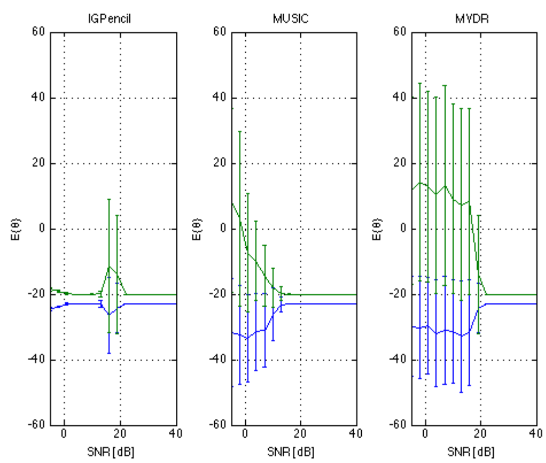

Fig. 3. Statistical performance for two sources at $[-20^o, -23^o]$ by MUSIC, IGPencil and MVDR for SNR ranging from -5 to 40dB

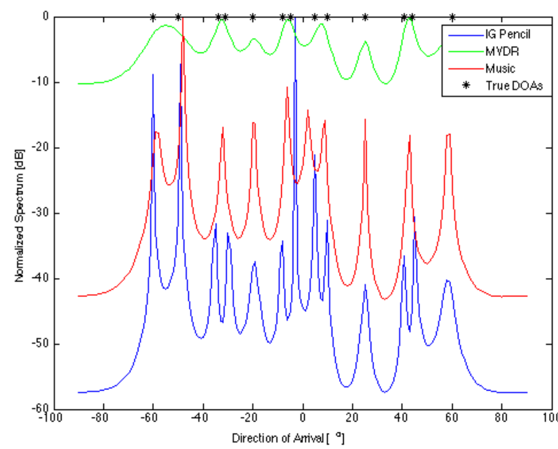

Fig. 5. Comparison between IGPencil, MUSIC and MVDR spectrum for 13 sources, 11 antenna elements and an SNR of 10dB

6. CONCLUSIONS

In this paper, a new direction of arrival estimation approach is proposed based on distance notions taken from IG. By describing a probability distribution a structure of differential geometry and defining a statistical manifold parametrized by a covariance matrix, it is possible to assess how close the sample covariance matrix is from a given covariance matrix. The proposed method exploits these geodesic distances in order to formulate the DOA estimation problem as an optimization framework. The optimization problem is reduced to a line search when the feasible set is selected as $\mathcal{A}_1$. The relation between the MVDR beamformer and IGPencil is elucidated. Simulation results have illustrated the performance of the proposed method. A comparison between MUSIC, MVDR and IGPencil has shown that the proposed method provides an equivalent performance at high SNR. At low SNR the nature of the method provides an improvement in resolution capabilities. Exploring this method for the case of different array topologies as well as model selection based on information geometry is currently a topic of future investigations.

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