A REFERENCE-FREE TIME DIFFERENCE OF ARRIVAL SOURCE LOCALIZATION USING A PASSIVE SENSOR ARRAY

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ABSTRACT

Least squares source position estimation techniques from time difference of arrival measurements are based on choosing a reference sensor. Selecting different reference sensors may affect the positioning accuracy by a considerable amount. We suggest a closed-form least squares position estimation using all the available distinct time differences, which does not involve the selection of a reference sensor. The nonlinear terms, associated with the distances between the sensors and the source, are eliminated with an orthogonal projection matrix. Simulation results show that the proposed approach outperforms previous closed-form least squares solutions.

1. INTRODUCTION

Passive source localization has been under study for many years, and has found various applications in radar, sonar, wireless communications, underwater acoustics, and sensor networks. One common technique is based on measuring the time difference of arrival (TDOA) of the source signal to several spatially distributed receivers.

The source position is determined from the intersection of a set of hyperbolic equations defined by the TDOA estimates. A straightforward approach to solve this nonlinear problem is to use the maximum likelihood estimator (MLE), which under regularity conditions is asymptotically unbiased, and approaches the Cramer Rao lower bound (CRLB). The main difficultly is that the MLE is computationally intensive since it requires a two or three (depending on the problem geometry) dimensional search over the position space.

To overcome this difficulty, other approaches were suggested in the literature to solve these non-linear hyperbolic equations [1–7]. One approach is to linearize the hyperbolic equations using Taylor series [3, 4] which is an iterative approach that requires an initial guess. Another approach is to reorganize the nonlinear hyperbolic equations into a set of linear equations, and then estimate the position with a least squares (LS) method [5–7]. The idea is to define the distance between the reference sensor and the source as an unknown nuisance parameter. The two-step estimation method in [5] first estimates the source position and this nuisance parameter using an unconstrained LS method. Then, the position is estimated by solving another LS problem where the constraint on the nuisance parameter is taken into account. The method in [6] is based on expressing the source position in terms of the nuisance parameter, and obtains its solution using LS. On the other hand, the method in [7] first eliminates the nuisance parameter with an orthogonal projection matrix, and then solves the source position using LS. The last two solutions were shown to be mathematically equivalent [7]. All of these solutions are based on selecting a reference sensor. In [7, Section VI] it was mentioned that choosing different reference sensors affects the positioning accuracy by a considerable amount.

Assuming a sensor array with M sensors, for each selection of a reference sensor, we collect $M-1$ TDOA measurements. However, the maximum number of distinct TDOA measurements is $(M-1)M/2$. The latter is referred to as the full set. Recently, a technique to obtain the optimal non-redundant set of $M-1$ TDOA measurements with respect to a reference sensor out of the full set was proposed in [8]. The conclusion was that the optimal non-redundant set should be used instead of the full set to reduce the complexity load. However, the model in [8] assumes that the observed signal by each sensor is only time delayed but not attenuated. Since the sensors are spatially distributed, the attenuation of each observed signal depends on the position of the source (as is usually modeled in free space propagation). As a result, the conversion from the full set to the optimal non-redundant set depends on the unknown position as well.

Herein, a reference-free TDOA (RF-TDOA) based positioning technique is proposed. By "reference-free" we mean a positioning technique which does not involve the selection of a reference point. The idea is to estimate the position using all the available TDOAs. This leads to a set of equations which depends on the unknown source position and a set of nuisance parameters, where each nuisance parameter is the distance between a sensor position and the source position. These nuisance parameters are eliminated by a projection onto the orthogonal complement of the span of the matrix that contains all the possible TDOAs. The source position is then estimated with a LS method. Simulations demonstrate that the proposed approach outperforms the method in [7] for any choice of a reference sensor. It is worth noting that a weighted LS solution was also obtained in [7]. However, the weighting matrix depends on the unknown position of the source, and the LS solution is usually used in such a case as a starting point of the weighted LS approach. This is the reason for our focus on the LS solution.

2. PROBLEM FORMULATION

2.1. The model

Consider M spatially distributed receivers and a source, where all are located in a plane (the extension to a three dimensional space is straightforward). The source radiates an isotropic narrowband complex envelope at a carrier frequency. Since the radiated wave is spherical, the waveform impinges on the different receivers with different delays and attenuations. Assuming a free space propagation, the low-pass equivalent discrete-time signal observed by the

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mth receiver is therefore expressed as

$$r_m(n) = \frac{\kappa}{\rho_m} s(n - \tau_m) + e_m(n) \, , n = 0, \ldots, N-1 \quad (1)$$

where N is the number of samples, $s(n)$ is the low-pass equivalent discrete-time source signal, κ is a constant [9], and $\tau_m = \rho_m / c$ is the propagation time of the signal with c the signal propagation speed, and $\rho_m \triangleq \|\mathbf{p}_m - \mathbf{p}_s\|$ the distance between the mth sensor and the source. Note that $\mathbf{p}_m = [x_m, y_m]^T$, $m = 1, 2, ..., M$, and $\mathbf{p}_s = [x_s, y_s]^T$ denote the coordinates of the mth sensor and the source, respectively. Also, $\{e_m(n)\}_{m=1,n=0}^{M,N-1}$ are additive noises at the output of the receivers, due to thermal noise. We assume that $\{e_m(n)\}_{m=1,n=0}^{M,N-1}$ are zero-mean white Gaussian processes with variance σ_e^2, independent of $s(n)$, and that $s(n)$ is also a zero-mean white Gaussian process with variance σ_s^2 [8].

2.2. The range-difference measurements

Let $\tau_{m,k} \triangleq \tau_m - \tau_k$ be the TDOA between the mth and kth signals, and the range difference associated with it be

$$d_{m,k} \triangleq c\tau_{m,k} = \rho_m - \rho_k \quad (2)$$

The estimate of $\tau_{m,k}$, denoted by $\hat{\tau}_{m,k}$, is obtained by cross-correlating two signals, that is [8],

$$\hat{\tau}_{m,k} = \underset{\tau_{m,k}}{\text{argmax}} \left\{ \sum_{n=0}^{N-1} r_m(n) r_k(n - \tau_{m,k}) \right\} \quad (3)$$

The range difference estimate is then given by $\hat{d}_{m,k} = c\hat{\tau}_{m,k}$. Note that $\hat{d}_{m,k} = -\hat{d}_{k,m}$ and therefore there are only $M(M-1)/2$ distinct range measurements (known as the full set of measurements). Similarly to [8] we define the vector containing all the distinct TDOA estimates

$$\hat{\mathbf{d}} \triangleq [\hat{d}_{2,1}, \ldots, \hat{d}_{M,1}, \hat{d}_{3,2}, \ldots, \hat{d}_{M,2}, \ldots, \hat{d}_{M,M-1}]^T \quad (4)$$

We assume that

$$\hat{d}_{m,k} = d_{m,k} + \eta_{m,k} \quad (5)$$

where $\eta_{m,k}$ are zero mean Gaussian random variables. The difference between the current modeling and that in [8] is expressed through the matrix $\text{cov}(\hat{\mathbf{d}}, \hat{\mathbf{d}})$. In [8, Appendix I] the variance of $\hat{d}_{m,n}$ and the covariance between $\hat{d}_{i,j}$ and $\hat{d}_{k,\ell}$ are derived assuming that the attenuations of the signals observed by all receivers are identical. By applying the model in (1) to the analysis in [8, Appendix I] we get[1]

$$\text{cov}(\hat{d}_{m,n}, \hat{d}_{m,n}) = \frac{3c^2}{\pi^2 N \kappa^2} \frac{\rho_m \rho_n}{\text{SNR}} \left(2 + \frac{\rho_m \rho_n}{\text{SNR}}\right) \quad (6)$$

$$\text{cov}(\hat{d}_{i,j}, \hat{d}_{k,\ell}) = \frac{3c^2}{\pi^2 N \kappa^2} \frac{\rho_i \rho_j \rho_k \rho_\ell}{\text{SNR}^2} \phi_{i,j,k,\ell} \quad (7)$$

where $\phi_{i,j,k,\ell}$ is defined as [8, Eq. (6)],

$$\phi_{i,j,k,\ell} = \begin{cases} 1 & , i = k \text{ and } j \neq \ell, i \neq k \text{ and } j = \ell \\ -1 & , i = \ell \text{ and } j \neq k, i \neq \ell \text{ and } j = k \\ 0 & , j \neq k \neq i \neq \ell \end{cases} \quad (8)$$

[1] Notice that if the signal is observed by all sensors with identical attenuations we get [8, Eq. (5)-(6)].

and the signal to noise ratio (SNR) is defined as,

$$\text{SNR} \triangleq \frac{\sigma_s^2}{\sigma_e^2} \quad (9)$$

The problem is to determine the position of the source given the estimates $\{\hat{d}_{m,k}\}_{m,k=1}^{M}$.

3. THE PROPOSED POSITIONING TECHNIQUE

We start by considering the noise-less case. Note that we can write the square of ρ_k as

$$\rho_k^2 = (\rho_m + d_{k,m})^2 = \rho_m^2 + 2\rho_m d_{k,m} + d_{k,m}^2 \quad (10)$$

Define the $M \times 1$ vectors

$$\mathbf{d}_m \triangleq [d_{1,m}, \ldots, 0, \ldots, d_{M,m}]^T \quad (11)$$

$$\boldsymbol{\rho} \triangleq [\rho_1, \ldots, \rho_M]^T \quad (12)$$

where the zero in the vector $\mathbf{d}_m$ is in the mth entry. We can rewrite (10) in a vector form as

$$\boldsymbol{\rho} \odot \boldsymbol{\rho} = \mathbf{1}\rho_m^2 + 2\mathbf{d}_m \rho_m + \mathbf{d}_m \odot \mathbf{d}_m \quad (13)$$

where $\odot$ is the hadamard product, and $\mathbf{1}$ is a $M \times 1$ vector with all entries equal to unity. Define the $M \times M$ matrix $\mathbf{D}$ that contains all the range differences,

$$\mathbf{D} \triangleq [\mathbf{d}_1, \cdots, \mathbf{d}_M] \quad (14)$$

Using (14) we can now further write (13) in a matrix form as

$$(\boldsymbol{\rho} \odot \boldsymbol{\rho})\mathbf{1}^T = \mathbf{1}(\boldsymbol{\rho} \odot \boldsymbol{\rho})^T + 2\mathbf{D} \odot (\mathbf{1}\boldsymbol{\rho}^T) + \mathbf{D} \odot \mathbf{D} \quad (15)$$

Multiplying (15) from the right by $\mathbf{1}$ and dividing by $2M$ yields

$$\frac{1}{2M}(\mathbf{D} \odot \mathbf{D})\mathbf{1} + \frac{1}{M}\mathbf{D}\boldsymbol{\rho} = \frac{1}{2}\mathbf{S}(\boldsymbol{\rho} \odot \boldsymbol{\rho}) \quad (16)$$

where we define the $M \times M$ orthogonal projection matrix,

$$\mathbf{S} \triangleq \mathbf{I} - \frac{1}{M}\mathbf{1}\mathbf{1}^T \quad (17)$$

We further simplify (16) by noticing that the mth element of $\mathbf{S}(\boldsymbol{\rho} \odot \boldsymbol{\rho})$ can be written as

$$\rho_m^2 - \frac{1}{M}\sum_{n=1}^{M} \rho_n^2 = \|\mathbf{p}_m\|^2 - \frac{1}{M}\sum_{n=1}^{M} \|\mathbf{p}_n\|^2 + 2(\frac{1}{M}\mathbf{P}\mathbf{1} - \mathbf{p}_m)^T \mathbf{p}_s \quad (18)$$

where we define the $2 \times M$ matrix

$$\mathbf{P} \triangleq [\mathbf{p}_1, \cdots, \mathbf{p}_M] \quad (19)$$

which contains the positions of all sensors. Using (18) we can now express $\mathbf{S}(\boldsymbol{\rho} \odot \boldsymbol{\rho})$ as

$$\mathbf{S}(\boldsymbol{\rho} \odot \boldsymbol{\rho}) = \mathbf{S}\mathbf{u} - 2\mathbf{S}\mathbf{P}^T \mathbf{p}_s \quad (20)$$

where we define the $M \times 1$ vector

$$\mathbf{u} \triangleq [\|\mathbf{p}_1\|^2, \ldots, \|\mathbf{p}_M\|^2]^T \quad (21)$$

which includes the squared norms of the positions. Substituting (20) into (16) yields,

$$\frac{1}{2M}(\mathbf{D}\odot\mathbf{D})\mathbf{1}+\frac{1}{M}\mathbf{D}\boldsymbol{\rho}=\frac{1}{2}\mathbf{S}\mathbf{u}-\mathbf{S}\mathbf{P}^T\mathbf{p}_s \tag{22}$$

This result depends linearly on the parameter of interest $\mathbf{p}_s$, but also on the nuisance vector $\boldsymbol{\rho}$. The idea is to eliminate the nuisance term $\mathbf{D}\boldsymbol{\rho}$ in (22) by pre-multiplying (22) with a matrix $\mathbf{G}$ such that $\mathbf{G}\mathbf{D} = \mathbf{0}$, and then (22) becomes a linear equation with respect (w.r.t.) to the unknown position $\mathbf{p}_s$. To obtain this matrix we use the following proposition.

Claim 1. $rank(\mathbf{D}) = 2$ *with probability one.*

Proof. Observe that we can write the matrix in (14) as

$$\mathbf{D}=\boldsymbol{\rho}\mathbf{1}^T-\mathbf{1}\boldsymbol{\rho}^T=\mathbf{B}_1\mathbf{B}_2 \tag{23}$$

where $\mathbf{B}_1 \triangleq [\boldsymbol{\rho},\mathbf{1}]$, and $\mathbf{B}_2 \triangleq [\mathbf{1},-\boldsymbol{\rho}]^T$. Notice that $rank(\mathbf{B}_1) = 2$ and $rank(\mathbf{B}_2) = 2$ unless $\forall m:\ \|\mathbf{p}_m-\mathbf{p}_s\| = \alpha$, where α is a scalar, that is, only if the sensors are located on a circle with a radius of α, and the source is located exactly in the center of the circle. Since the source can be located anywhere in the continuous position space, this situation occurs with probability zero. We therefore conclude that $rank(\mathbf{D}) = 2$ with probability 1. □

According to Claim 1 we can express the singular value decomposition (SVD) of $\mathbf{D}$ as

$$\mathbf{D}=[\mathbf{U},\mathbf{V}]\boldsymbol{\Lambda}[\mathbf{U},\mathbf{V}]^T \tag{24}$$

where $\mathbf{U}$ is a $M\times2$ matrix that contains the two eigenvectors associated with the two non-zero eigenvalues, $\mathbf{V}$ is a $M\times(M-2)$ matrix that contains the $M-2$ eigenvectors associated with the $M-2$ zero eigenvalues, and $\boldsymbol{\Lambda}$ is a $M\times M$ diagonal matrix where the first two elements on the diagonal are the two non-zero eigenvalues, and the other elements are zero. We therefore conclude that $\mathbf{G}=\mathbf{V}^T$. Pre-multiplying (22) by $\mathbf{V}^T$ results in

$$\mathbf{V}^T\mathbf{S}\mathbf{P}^T\mathbf{p}_s=\mathbf{z} \tag{25}$$

where the $(M-2)\times1$ vector $\mathbf{z}$ is defined as

$$\mathbf{z}\triangleq\frac{1}{2}\mathbf{V}^T\left(\mathbf{S}\mathbf{u}-\frac{1}{M}(\mathbf{D}\odot\mathbf{D})\,\mathbf{1}\right) \tag{26}$$

The result in (25) is a linear model w.r.t. the vector of interest $\mathbf{p}_s$. In case noise is present, $\mathbf{p}_s$ can be estimated with a LS method. The estimated position is then given as

$$\hat{\mathbf{p}}_s=\underset{\mathbf{p}_s}{\operatorname{argmin}}\{\|\mathbf{V}^T\mathbf{S}\mathbf{P}^T\mathbf{p}_s-\mathbf{z}\|^2\}=\mathbf{Q}\mathbf{z} \tag{27}$$

where we define the $2\times(M-2)$ matrix $\mathbf{Q}$ as

$$\mathbf{Q}\triangleq\left(\mathbf{P}\mathbf{S}\mathbf{V}\mathbf{V}^T\mathbf{S}\mathbf{P}^T\right)^{-1}\mathbf{P}\mathbf{S}\mathbf{V} \tag{28}$$

This concludes the proposed positioning technique.

4. THE CRAMÉR-RAO LOWER BOUND

The CRLB is a lower bound on any unbiased estimator of $\mathbf{p}_s$, that is, $\text{cov}(\hat{\mathbf{p}}_s)\geq\mathbf{J}^{-1}(\mathbf{p}_s)$, where $\mathbf{J}(\mathbf{p}_s)$ is the 2×2 Fisher information matrix (FIM). The FIM of the position vector is expressed in [8]. However, due to the use of the attenuation in modeling the received signal, the result expressed in [8, Eq. (24)] is slightly modified, and is given as

$$\mathbf{J}(\mathbf{p}_s)=\mathbf{H}\text{cov}(\hat{\mathbf{d}},\hat{\mathbf{d}})^{-1}\mathbf{H}^T \tag{29}$$

where

$$\mathbf{H} \triangleq [\mathbf{h}_{2,1},\ldots,\mathbf{h}_{M,1},\mathbf{h}_{3,2},\ldots,\mathbf{h}_{M,M-1}]^T \tag{30}$$

$$\mathbf{h}_{m,n} \triangleq \frac{1}{\rho_m}(\mathbf{p}_s-\mathbf{p}_m)-\frac{1}{\rho_n}(\mathbf{p}_s-\mathbf{p}_n) \tag{31}$$

and the entries of $\text{cov}(\hat{\mathbf{d}},\hat{\mathbf{d}})$ are given in (6)-(7).

5. NUMERICAL RESULTS

We demonstrate the empirical position root mean square error (PRMSE) of the proposed RF-TDOA algorithm, and compare it with the closed-form LS solution suggested in [7]. We define the PRMSE as PRMSE $\triangleq \sqrt{\frac{1}{N_{exp}}\sum_{i=1}^{N_{exp}}\|\mathbf{p}_s-\hat{\mathbf{p}}_{s,i}\|^2}$, where $\hat{\mathbf{p}}_{s,i}$ is the estimated source position at the ith Monte-Carlo (MC) trial, and $N_{exp}=1000$. As a benchmark we also compute the theoretical PRMSE obtained from using the CRLB in (29).

In the first simulation, we examined the PRMSEs versus SNR. We consider six sensors ($M = 6$), where the position of the mth sensor is $\mathbf{p}_m = 1000\cdot[\cos\left(\frac{2\pi m}{M}\right),\sin\left(\frac{2\pi m}{M}\right)]^T$ [Meter], $m = 1,\ldots,6$. We consider two source positions: i) Far-field region - the source position is $\mathbf{p}_s=[7000,3000]^T$ [Meter]; ii) Near-field region - the source position is $\mathbf{p}_s=[2000,3000]^T$ [Meter]. We varied the SNR from 0 [dB] to 20 [dB] with a step of 2 [dB]. In Figure 1 we show the PRMSE of the proposed RF-TDOA method, and the PRMSE of the method in [7] (for different reference selections) versus $\frac{\kappa^2}{c^2}$SNR. As can be seen, the RF-TDOA method outperforms the method in [7] for any selection of the reference sensor. For a source in the far-field region, the RF-TDOA method also achieves the CRLB.

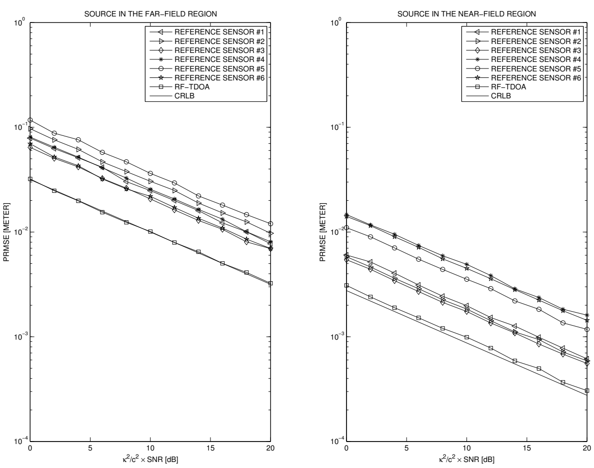

Fig. 1. PRMSEs of the proposed method, the method in [7] (for different selections of the reference sensor), and using the CRLB, versus $\frac{\kappa^2}{c^2}$SNR given a sensor array with a circular configuration.

In the second simulation, we examined the PRMSEs for random sensor configurations assuming the same source coordinates as detailed in the first simulation. We performed 50 random configurations. In each configuration the coordinates of each sensor were randomly and independently deployed according to a uniform distribution, i.e., $x_m \in$ Uniform $[-1000, 1000]$ [Meter] and $y_m \in$ Uniform $[-1000, 1000]$ [Meter]. We then averaged the PRMSEs over the 50 random configurations. In Figure 2 we show the PRMSE of the proposed RF-TDOA method, and the PRMSE of the method in [7] (for different reference selections) versus $\frac{\kappa^2}{c^2}$SNR. Again, the RF-TDOA method outperforms the method in [7] for any reference selection.

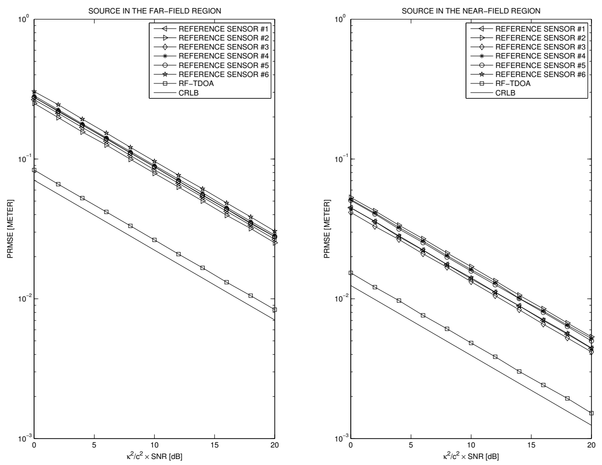

Fig. 2. PRMSEs of the proposed method, the method in [7] (for different selections of the reference sensor), and using the CRLB, versus $\frac{\kappa^2}{c^2}$SNR given a sensor array with a random configuration.

Finally, in the third simulation, the position of the mth sensor is as detailed in the first simulation. We examined the PRMSEs versus the source position. We changed the position of the source as $\mathbf{p}_s = 7000 \cdot [\cos(\phi), \sin(\phi)]^T$ [Meter] where ϕ is varied from 2 [Deg] to 338 [Deg] with a step of 12 [Deg]. We set $\frac{\kappa^2}{c^2}\text{SNR} = 10$[dB]. The PRMSEs results of the RF-TDOA method and the method in [7] are shown in Figure 3. The RF-TDOA method has a better PRMSE performance than the method in [7] for any selection of the reference sensor and any position of the source.

6. CONCLUSION

The accuracy of source positioning techniques given TDOA measurements depends on the choice of the reference sensor. Herein, we suggested a closed-form LS solution to the problem which does not involve the selection of a reference point. Simulations demonstrated that the PRMSE of the proposed estimator is better than that of the previous closed-form LS solutions. Ongoing research is focused on: i) developing a weighted LS solution; ii) analyzing the asymptotic bias and covariance matrix; iii) evaluating the computational complexity of the method, and iv) comparing the method with the two-step positioning approach [5].

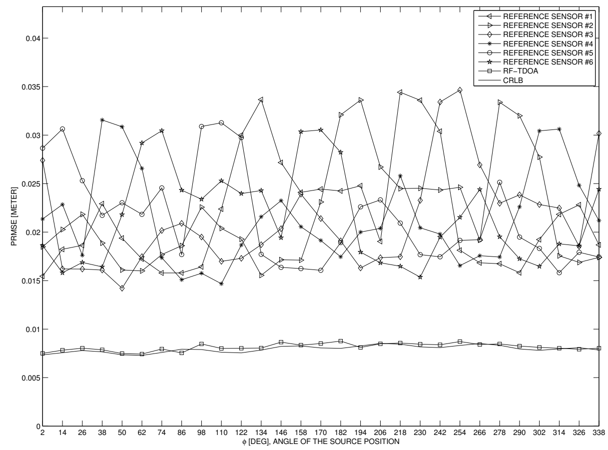

Fig. 3. PRMSEs of the proposed method, the method in [7] (for different selections of the reference sensor), and using the CRLB, versus the angle of the source position given a sensor array with a circular configuration.

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