Electromagnetic Wave Propagation in a Time-Variant Medium: A State-Space Approach Via the Spatial Fourier Domain

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Abstract — **In this paper we consider a three-dimensional electromagnetic field in a homogeneous and time-variant medium. After a spatial Fourier transformation we solve Maxwell's equations for the electric and magnetic flux densities and write the solution in terms of the so-called transition matrix. This matrix can be given in terms of the well known Peano-Baker series and we show that in the constant impedance case this series simplifies to the matrix exponential operator. This operator can explicitly be transformed back to the spatial domain resulting in explicit space-time domain expression for the electromagnetic field.**

1 INTRODUCTION

Interactions of the electromagnetic field with a time-varying medium are exploited in many different application areas such as imaging, radar, telecommunications, and optical communication technology ([1] – [4]). In this paper we study this interaction for a three-dimensional electromagnetic field that is present in a homogeneous and time-variant medium. The field is generated by external electric- and magnetic-current sources that occupy a bounded domain $\mathbb{D}^{\rm src}$ in $\mathbb{R}^3$. These sources are switched on at the time instant $t = t_0$ and it is our objective to find the electric and magnetic field strengths in all of space for $t > t_0$.

Our approach is to solve Maxwell's equations for the flux densities instead of the field strengths. Specifically, we first use the constitutive relations to rewrite Maxwell's equations in terms of the electric and magnetic fluxes. We then apply a three-dimensional spatial Fourier transform to arrive at the Fourier domain state-space representation of Maxwell's equations. From this representation we obtain the transformed electric and magnetic flux densities in terms of the so-called Peano-Baker series. The problem with this series is that it is far from trivial to analytically evaluate its inverse spatial Fourier transform. If, however, the transformed Maxwell operator commutes with its time-integrated form, then the Peano-Baker series simplifies to the exponent of the time-integrated Maxwell operator and we will show that under this condition it is possible to analytically carry out the inverse Fourier transform. Furthermore, we will also show that the commutation relation holds if and only if the time variations of our medium are such that its impedance is constant. Having found the electric and magnetic flux densities in space-time, it is now an easy matter to obtain the field strengths by simply using the constitutive relations one more time.

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2 GENERAL k-SPACE SOLUTION

We consider an electromagnetic field in a homogeneous and time-variant medium characterized by the time-dependent permittivity $\varepsilon(t)$ and permeability $\mu(t)$. The electromagnetic field is generated by external electric- and magnetic-current sources $J_k^{\rm ext}$ and $K_j^{\rm ext}$ that occupy the bounded domain $\mathbb{D}^{\rm src}$ in $\mathbb{R}^3$. The sources start to act at the time instant $t = t_0$ and we are interested in the electromagnetic field in all of space and for $t \geq t_0$. This field satisfies Maxwell's equations

$$-\epsilon_{k,m,p}\partial_m H_p + \partial_t D_k = -J_k^{\rm ext} \tag{1}$$

and

$$\epsilon_{j,m,r}\partial_m E_r + \partial_t B_j = -K_j^{\rm ext} \tag{2}$$

and the constitutive relations relating the fluxes to the fields are given by

$$D_k = \varepsilon(t)E_k \quad \text{and} \quad B_j = \mu(t)H_j. \tag{3}$$

Our approach is to first solve Maxwell's equations for the fluxes D_k and B_j and then to use the above constitutive relations to obtain the corresponding electric and magnetic field strength components. Explicitly, we write Maxwell's equations as

$$-\mu(t)^{-1}\epsilon_{k,m,p}\partial_m B_p + \partial_t D_k = -J_k^{\rm ext} \tag{4}$$

and

$$\varepsilon(t)^{-1}\epsilon_{j,m,r}\partial_m D_r + \partial_t B_j = -K_j^{\rm ext}, \tag{5}$$

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and try to solve these equations for D_k and B_j using the prescribed initial fluxes

$$D_k = D_k^0 \quad \text{and} \quad B_j = B_j^0 \tag{6}$$

which hold for all $\mathbf{x} \in \mathbb{R}^3$ and at $t = t_0$.

To this end, we first apply a Fourier transformation to Maxwell's equations with respect to the spatial coordinates. The Fourier transformation pair of a function $f(\mathbf{x}, t)$ is given by

$$\begin{aligned} \tilde{f}(\mathbf{k}, t) &= \mathcal{F}\{f(\mathbf{x}, t)\} \\ &= \int_{\mathbf{x}\in\mathbb{R}^3} \exp(\mathrm{j}\mathbf{k}\cdot\mathbf{x}) f(\mathbf{x}, t)\,\mathrm{dV} \end{aligned} \tag{7}$$

and

$$\begin{aligned} f(\mathbf{x}, t) &= \mathcal{F}^{-1}\{\tilde{f}(\mathbf{k}, t)\} \\ &= \frac{1}{(2\pi)^3} \int_{\mathbf{k}\in\mathbb{R}^3} \exp(-\mathrm{j}\mathbf{k}\cdot\mathbf{x}) \tilde{f}(\mathbf{k}, t)\,\mathrm{dV}, \end{aligned} \tag{8}$$

and using the rule that $\tilde{\partial}_m = -\mathrm{j}k_m$, we arrive at

$$\partial_t \tilde{D}_k = \mu(t)^{-1} \epsilon_{k,m,p} (-\mathrm{j}k_m) \tilde{B}_p - \tilde{J}_k^{\mathrm{ext}} \tag{9}$$

and

$$\partial_t \tilde{B}_j = -\varepsilon(t)^{-1} \epsilon_{j,m,r} (-\mathrm{j}k_m) \tilde{D}_r - \tilde{K}_j^{\mathrm{ext}}. \tag{10}$$

We now write these two equations in state-space form. Introducing the $\mathbf{k}$-space flux and source vectors as

$$\tilde{\mathbf{f}} = \begin{pmatrix} \tilde{\mathbf{d}} \\ \tilde{\mathbf{b}} \end{pmatrix} \quad \text{and} \quad \tilde{\mathbf{q}} = \begin{pmatrix} \tilde{\mathbf{j}}^{\mathrm{ext}} \\ \tilde{\mathbf{k}}^{\mathrm{ext}} \end{pmatrix}, \tag{11}$$

respectively, where

$$\tilde{\mathbf{d}} = \begin{pmatrix} \tilde{D}_1 \\ \tilde{D}_2 \\ \tilde{D}_3 \end{pmatrix}, \quad \tilde{\mathbf{b}} = \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \\ \tilde{B}_3 \end{pmatrix}, \tag{12}$$

$$\tilde{\mathbf{j}}^{\mathrm{ext}} = \begin{pmatrix} \tilde{J}_1^{\mathrm{ext}} \\ \tilde{J}_2^{\mathrm{ext}} \\ \tilde{J}_3^{\mathrm{ext}} \end{pmatrix} \quad \text{and} \quad \tilde{\mathbf{k}}^{\mathrm{ext}} = \begin{pmatrix} \tilde{K}_1^{\mathrm{ext}} \\ \tilde{K}_2^{\mathrm{ext}} \\ \tilde{K}_3^{\mathrm{ext}} \end{pmatrix}, \tag{13}$$

we can write

$$\partial_t \tilde{\mathbf{f}} = \tilde{\mathbf{A}}(t)\tilde{\mathbf{f}} - \tilde{\mathbf{q}}, \tag{14}$$

where $\tilde{\mathbf{A}}(t)$ is the $\mathbf{k}$-space system matrix given by

$$\tilde{\mathbf{A}}(t) = \begin{pmatrix} \mathbf{0} & \mu(t)^{-1}\tilde{\mathbf{K}} \\ -\varepsilon(t)^{-1}\tilde{\mathbf{K}} & \mathbf{0} \end{pmatrix} \tag{15}$$

and

$$\tilde{\mathbf{K}} = \begin{pmatrix} 0 & \mathrm{j}k_3 & -\mathrm{j}k_2 \\ -\mathrm{j}k_3 & 0 & \mathrm{j}k_1 \\ \mathrm{j}k_2 & -\mathrm{j}k_1 & 0 \end{pmatrix} \tag{16}$$

is the Fourier transformed curl operator. From linear system theory we know that the general $\mathbf{k}$-space solution of Eq. (14) is given by (see, for example, [5])

$$\tilde{\mathbf{f}}(t) = \tilde{\mathbf{\Phi}}(t, t_0)\tilde{\mathbf{f}}_0 - \int_{\tau=t_0}^{t} \tilde{\mathbf{\Phi}}(t, \tau)\tilde{\mathbf{q}}(\tau)\,\mathrm{d}\tau \quad \text{for } t \geq t_0, \tag{17}$$

where $\tilde{\mathbf{f}}_0$ contains the Fourier-transformed initial fluxes and $\tilde{\mathbf{\Phi}}$ is the transition matrix given by the Peano-Baker series

$$\begin{aligned} \tilde{\mathbf{\Phi}}(t, \tau) = \mathbf{I} &+ \int_{\sigma_1=\tau}^{t} \tilde{\mathbf{A}}(\sigma_1)\,\mathrm{d}\sigma_1 \\ &+ \int_{\sigma_1=\tau}^{t} \tilde{\mathbf{A}}(\sigma_1) \int_{\sigma_2=\tau}^{\sigma_1} \tilde{\mathbf{A}}(\sigma_2)\,\mathrm{d}\sigma_2\,\mathrm{d}\sigma_1 + \dots . \end{aligned} \tag{18}$$

Obtaining an explicit (space-time) expression for the transition matrix is difficult and perhaps even impossible in general. In the next section, however, we show that the above expression for the transition matrix simplifies significantly if the impedance is constant in time.

3 EXPLICIT SPACE-TIME SOLUTION FOR MEDIA WITH A CONSTANT IMPEDANCE

Suppose that the time variation of the permittivity and permeability is such that the system matrix commutes with its time-integrated form. More precisely, suppose that for every value of τ and t we have

$$\tilde{\mathbf{A}}(t) \int_{\sigma=\tau}^{t} \tilde{\mathbf{A}}(\sigma)\,\mathrm{d}\sigma = \int_{\sigma=\tau}^{t} \tilde{\mathbf{A}}(\sigma)\,\mathrm{d}\sigma\,\mathbf{A}(t). \tag{19}$$

It is well known that if this commutation relation holds the expression for the transition matrix reduces to (see [5])

$$\tilde{\mathbf{\Phi}}(t, \tau) = \exp\left[\int_{\sigma=\tau}^{t} \tilde{\mathbf{A}}(\sigma)\,\mathrm{d}\sigma\right]. \tag{20}$$

This form allows us to obtain explicit expressions for the electromagnetic field quantities in space-time. Before doing so, however, we first want to determine under what condition(s) on ε and μ the commutation relation of Eq. (19) holds. Writing out the left- and right-hand side of this equation and comparing entries, we find that the commutation relation is satisfied if and only if

$$Z(t)^2 \int_{\sigma=\tau}^{t} \frac{1}{\mu(\sigma)}\,\mathrm{d}\sigma = \int_{\sigma=\tau}^{t} Z(\sigma)^2 \frac{1}{\mu(\sigma)}\,\mathrm{d}\sigma, \tag{21}$$

for every value of τ and t, where $Z = [\mu(t)/\varepsilon(t)]^{1/2}$ is the wave impedance. Since this equation is satisfied for a constant impedance only, we conclude that the simplified form for the transition matrix as given by Eq. (20) holds if and only if the time variations of the permittivity and permeability are such that the impedance is constant.

From this moment on, we restrict ourselves to this constant impedance case and write the system matrix as

$$\tilde{\mathbf{A}} = c(t)\tilde{\mathbf{B}}, \tag{22}$$

where $c = [\varepsilon(t)\mu(t)]^{-1/2}$ is the electromagnetic wave speed and

$$\tilde{\mathbf{B}} = \begin{pmatrix} \mathbf{0} & Z^{-1}\tilde{\mathbf{K}} \\ -Z\tilde{\mathbf{K}} & \mathbf{0} \end{pmatrix} \tag{23}$$

is time independent. In terms of this matrix, the transition matrix is given by

$$\tilde{\boldsymbol{\Phi}}(t,\tau) = \exp\left[\zeta(t,\tau)\tilde{\mathbf{B}}\right], \tag{24}$$

where

$$\zeta(t,\tau) = \int_{\sigma=\tau}^{t} c(\sigma)\,\mathrm{d}\sigma. \tag{25}$$

We now rewrite the expression for the transition matrix in a form that allows for a direct transformation back to the space-time domain. First, we use the power series expansion of the exponential function and split the summation into even and odd powers of matrix $\tilde{\mathbf{B}}$. This gives

$$\tilde{\boldsymbol{\Phi}}(t,\tau) = \mathbf{I} + \sum_{n=1}^{\infty} \frac{\zeta^{2n}}{(2n)!}\tilde{\mathbf{B}}^{2n} + \sum_{n=0}^{\infty} \frac{\zeta^{2n+1}}{(2n+1)!}\tilde{\mathbf{B}}^{2n+1}. \tag{26}$$

Now since

$$\tilde{\mathbf{B}}^{2n} = (-1)^n|\mathbf{k}|^{2n} \begin{pmatrix} \tilde{\mathbf{G}} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{G}} \end{pmatrix} \quad \text{for } n = 1, 2, ..., \tag{27}$$

where $\tilde{\mathbf{G}} = \mathbf{I} - |\mathbf{k}|^{-2}\mathbf{k}\mathbf{k}^T$, and

$$\tilde{\mathbf{B}}^{2n+1} = (-1)^n|\mathbf{k}|^{2n}\tilde{\mathbf{B}} \quad \text{for } n = 0, 1, ..., \tag{28}$$

we can write Eq. (26) as

$$\begin{aligned} \tilde{\boldsymbol{\Phi}}(t,\tau) = \mathbf{I} &- \begin{pmatrix} \tilde{\mathbf{G}} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{G}} \end{pmatrix} \\ &+ \sum_{n=0}^{\infty}(-1)^n \frac{(\zeta|\mathbf{k}|)^{2n}}{(2n)!} \begin{pmatrix} \tilde{\mathbf{G}} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{G}} \end{pmatrix} \\ &+ \sum_{n=0}^{\infty}(-1)^n \frac{(\zeta|\mathbf{k}|)^{2n+1}}{(2n+1)!}|\mathbf{k}|^{-1}\tilde{\mathbf{B}}. \end{aligned} \tag{29}$$

In the above expressions we recognize the power series expansions of $\cos(\zeta|\mathbf{k}|)$ and $\sin(\zeta|\mathbf{k}|)$ and the above expression for $\tilde{\boldsymbol{\Phi}}$ becomes

$$\begin{aligned} \tilde{\boldsymbol{\Phi}}(t,\tau) = \mathbf{I} &- \begin{pmatrix} \tilde{\mathbf{G}} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{G}} \end{pmatrix} + \cos(\zeta|\mathbf{k}|) \begin{pmatrix} \tilde{\mathbf{G}} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{G}} \end{pmatrix} \\ &+ \frac{\sin(\zeta|\mathbf{k}|)}{|\mathbf{k}|}\tilde{\mathbf{B}}. \end{aligned} \tag{30}$$

Finally, since

$$\begin{aligned} \mathbf{I} &- \begin{pmatrix} \tilde{\mathbf{G}} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{G}} \end{pmatrix} + \cos(\zeta|\mathbf{k}|) \begin{pmatrix} \tilde{\mathbf{G}} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{G}} \end{pmatrix} \\ &= \cos(\zeta|\mathbf{k}|)\mathbf{I} + \frac{[1-\cos(\zeta|\mathbf{k}|)]}{|\mathbf{k}|^2} \begin{pmatrix} \mathbf{k}\mathbf{k}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{k}\mathbf{k}^T \end{pmatrix} \end{aligned} \tag{31}$$

we arrive at

$$\begin{aligned} \tilde{\boldsymbol{\Phi}}(t,\tau) &= \cos(\zeta|\mathbf{k}|)\mathbf{I} \\ &+ \frac{[1-\cos(\zeta|\mathbf{k}|)]}{|\mathbf{k}|^2} \begin{pmatrix} \mathbf{k}\mathbf{k}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{k}\mathbf{k}^T \end{pmatrix} \\ &+ \frac{\sin(\zeta|\mathbf{k}|)}{|\mathbf{k}|} \begin{pmatrix} \mathbf{0} & -Z^{-1}\tilde{\mathbf{K}} \\ Z\tilde{\mathbf{K}} & \mathbf{0} \end{pmatrix}, \end{aligned} \tag{32}$$

where we have used the definition of matrix $\tilde{\mathbf{B}}$ as well.

Having found an explicit $\mathbf{k}$-space expression for the transition matrix, we now substitute this expression in Eq. (17) to obtain the $\mathbf{k}$-space flux densities. Taking $\tilde{\mathbf{f}}_0 = \mathbf{0}$ for simplicity, we obtain

$$\begin{aligned} &\tilde{\mathbf{d}}(\mathbf{k},t) = \\ &-\int_{\tau=t_0}^{t} \left[\cos(\zeta|\mathbf{k}|)\tilde{\mathbf{j}}^{\mathrm{ext}} + \frac{[1-\cos(\zeta|\mathbf{k}|)]}{|\mathbf{k}|^2}\mathbf{k}\mathbf{k}^T\tilde{\mathbf{j}}^{\mathrm{ext}} \right. \\ &\qquad \left. - Z^{-1}\frac{\sin(\zeta|\mathbf{k}|)}{|\mathbf{k}|}\tilde{\mathbf{K}}\tilde{\mathbf{k}}^{\mathrm{ext}}\right]\mathrm{d}\tau \end{aligned} \tag{33}$$

and

$$\begin{aligned} &\tilde{\mathbf{b}}(\mathbf{k},t) = \\ &-\int_{\tau=t_0}^{t} \left[\cos(\zeta|\mathbf{k}|)\tilde{\mathbf{k}}^{\mathrm{ext}} + \frac{[1-\cos(\zeta|\mathbf{k}|)]}{|\mathbf{k}|^2}\mathbf{k}\mathbf{k}^T\tilde{\mathbf{k}}^{\mathrm{ext}} \right. \\ &\qquad \left. + Z\frac{\sin(\zeta|\mathbf{k}|)}{|\mathbf{k}|}\tilde{\mathbf{K}}\tilde{\mathbf{j}}^{\mathrm{ext}}\right]\mathrm{d}\tau, \end{aligned} \tag{34}$$

for $t \geq t_0$. To obtain the space-time representations for the fluxes, we need to evaluate the inverse Fourier transform of the above $\mathbf{k}$-space equations. To this end, we make use of the results

$$\begin{aligned} &\mathcal{F}^{-1}\left\{\cos(\zeta|\mathbf{k}|)\right\} = \\ &\frac{1}{4\pi|\mathbf{x}|}\left[\delta^{(1)}(\zeta-|\mathbf{x}|) - \delta^{(1)}(\zeta+|\mathbf{x}|)\right], \end{aligned} \tag{35}$$

$$\mathcal{F}^{-1}\left\{\frac{\sin(\zeta|\mathbf{k}|)}{|\mathbf{k}|}\right\} = \frac{1}{4\pi|\mathbf{x}|}\left[\delta(\zeta-|\mathbf{x}|)-\delta(\zeta+|\mathbf{x}|)\right], \tag{36}$$

and

$$\partial_t\mathcal{F}^{-1}\left\{\frac{[1-\cos(\zeta|\mathbf{k}|)]}{|\mathbf{k}|^2}\right\} = \frac{c(t)}{4\pi|\mathbf{x}|}\left[\delta(\zeta-|\mathbf{x}|)-\delta(\zeta+|\mathbf{x}|)\right]. \tag{37}$$

In the above expressions the second delta function between the square brackets vanishes, since $\zeta > 0$ for $t > t_0$.

Carrying out the inverse Fourier transform, we arrive at the space-time flux densities

$$\begin{aligned}\mathbf{d}(\mathbf{x},t) = &-\frac{1}{c(t)}\partial_t\mathbf{a}(\mathbf{x},t,t_0)\\ &+\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\int_{\sigma=t_0}^{t} c(\sigma)\mathbf{a}(\mathbf{x},\sigma,t_0)\,\mathrm{d}\sigma\\ &-Z^{-1}\boldsymbol{\nabla}\times\mathbf{m}(\mathbf{x},t,t_0)\end{aligned} \tag{38}$$

and

$$\begin{aligned}\mathbf{b}(\mathbf{x},t) = &-\frac{1}{c(t)}\partial_t\mathbf{m}(\mathbf{x},t,t_0)\\ &+\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\int_{\sigma=t_0}^{t} c(\sigma)\mathbf{m}(\mathbf{x},\sigma,t_0)\,\mathrm{d}\sigma\\ &+Z\boldsymbol{\nabla}\times\mathbf{a}(\mathbf{x},t,t_0),\end{aligned} \tag{39}$$

for $t \geq t_0$ and $\mathbf{x} \in \mathbb{R}^3$ and where the electric and magnetic vector potentials are given by

$$\mathbf{a}(\mathbf{x},t,t_0) = \int_{\mathbf{x}'\in\mathbb{D}^{\rm src}}\int_{\tau=t_0}^{t}\frac{\delta\left[\zeta(t,\tau)-|\mathbf{x}-\mathbf{x}'|\right]}{4\pi|\mathbf{x}-\mathbf{x}'|}\mathbf{j}^{\rm ext}(\mathbf{x}',\tau)\,\mathrm{d}\tau\,\mathrm{dV} \tag{40}$$

and

$$\mathbf{m}(\mathbf{x},t,t_0) = \int_{\mathbf{x}'\in\mathbb{D}^{\rm src}}\int_{\tau=t_0}^{t}\frac{\delta\left[\zeta(t,\tau)-|\mathbf{x}-\mathbf{x}'|\right]}{4\pi|\mathbf{x}-\mathbf{x}'|}\mathbf{k}^{\rm ext}(\mathbf{x}',\tau)\,\mathrm{d}\tau\,\mathrm{dV}, \tag{41}$$

respectively. Finally, the electric and magnetic field strengths are obtained by using the constitutive relations one more time.

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