Journal of Computational Physics 230 (2011) 1382–1386

Contents lists available at ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

On the relation between FDTD and Fibonacci polynomials

Rob F. Remis

Circuits and Systems Group, Faculty of Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, Mekelweg 4, 2628 CD Delft, The Netherlands

ARTICLE INFO

Article history:
Received 24 May 2010
Received in revised form 12 September 2010
Accepted 4 November 2010
Available online 27 November 2010

Keywords:
Finite-Difference Time-Domain method
Fibonacci polynomials
Chebyshev polynomials of the second kind
Stability condition

ABSTRACT

In this paper we show that the Finite-Difference Time-Domain method (FDTD method) follows the recurrence relation for Fibonacci polynomials. More precisely, we show that FDTD approximates the electromagnetic field by Fibonacci polynomials in $\Delta t\mathbf{A}$, where Δt is the time step and $\mathbf{A}$ is the first-order Maxwell system matrix. By exploiting the connection between Fibonacci polynomials and Chebyshev polynomials of the second kind, we easily obtain the Courant-Friedrichs-Lewy (CFL) stability condition and we show that to match the spectral width of the system matrix, the time step should be chosen as large as possible, that is, as close to the CFL upper bound as possible.

1. Introduction

The Finite-Difference Time-Domain method (FDTD method) is a well known and popular solution method for Maxwell's equations. It can be considered as one of the workhorses of computational electromagnetics and it is used to solve a wide range of electromagnetic wave field problems in many different areas of physics and engineering (see, for example [1–4]). A partial history of FDTD techniques can be found in [3]. The method is based on the first-order Maxwell system and solves for both the electric and magnetic field strength. In its most basic form, the spatial coordinates are discretized on a staggered nonuniform tensor-product grid (Yee-grid) and the time coordinate is discretized using a leap-frog time discretization scheme. The resulting FDTD method is explicit and conditionally stable, that is, FDTD is stable if and only if the time step satisfies the Courant-Friedrichs-Lewy (CFL) stability condition.

In this paper we show that FDTD field approximations are Fibonacci polynomials in $\Delta t\mathbf{A}$, where Δt is the time step and $\mathbf{A}$ is the first-order Maxwell system matrix. Furthermore, by exploiting the connection between Fibonacci polynomials and Chebyshev polynomials of the second kind, we can easily derive the CFL upper bound for FDTD. We also show that FDTD essentially does not adapt itself to the spectrum of the system matrix. We can only adapt FDTD to the spectral width of the system matrix through a proper choice of the time step Δt. Specifically, we show that FDTD is matched to the spectral width of the system matrix if we take the time step as large as possible. This result also follows from a standard numerical dispersion analysis of FDTD. Dispersion is minimized by iterating with a time step as close to the CFL limit as possible [4].

This paper is organized as follows. In Section 2 we briefly review the finite-difference state-space representation of the full 3D Maxwell system and discuss some of its properties. The standard FDTD update equations for three-dimensional problems are reviewed as well. In Section 3 we show the connection between Fibonacci polynomials and FDTD and show how the CFL stability bound is obtained by exploiting the connection between Fibonacci polynomials and Chebyshev polynomials of

E-mail address: R.F.Remis@TUDelft.NL

0021-9991/$ - see front matter © 2010 Elsevier Inc. All rights reserved.
doi:10.1016/j.jcp.2010.11.009

the second kind. Furthermore, we also show how we can match FDTD to the spectral width of the system matrix. Finally, the conclusions can be found in Section 4.

2. The finite-difference state-space representation

After discretizing Maxwell's equations in space on a staggered possibly nonuniform Yee-grid ([3,4]), we obtain the so-called finite-difference state-space representation [5]

$$(\mathbf{D} + \mathbf{M}\partial_t)\mathbf{f} = -w(t)\mathbf{q}. \tag{1}$$

In this equation, $\mathbf{f} = [\mathbf{e}^T, \mathbf{h}^T]^T$ is the field vector and $\mathbf{e}$ and $\mathbf{h}$ contain all time-dependent finite-difference approximations of the electric and magnetic field strength, respectively. The total number of field approximations (unknowns) is denoted by n. Furthermore, the source vector is given by $\mathbf{q} = [(\mathbf{j}^{\text{ext}})^T, (\mathbf{k}^{\text{ext}})^T]^T$, where $\mathbf{j}^{\text{ext}}$ and $\mathbf{k}^{\text{ext}}$ are the time-independent external finite-difference electric and magnetic current density vectors, and the scalar time-dependent function $w(t)$ is called the source wavelet or source signature.

The discretized rotation operators are contained in the spatial differentiation matrix

$$\mathbf{D} = \begin{pmatrix} \mathbf{0} & \mathbf{D}_{\text{h}} \\ \mathbf{D}_{\text{e}} & \mathbf{0} \end{pmatrix}.$$

In this equation, $\mathbf{D}_{\text{h}}$ is the discretized rotation operator (including a minus sign) acting on the magnetic field strength, while $\mathbf{D}_{\text{e}}$ is the discretized rotation operator acting on the electric field strength. Matrix $\mathbf{D}$ is skew-symmetric with respect to a diagonal and positive definite step size matrix $\mathbf{W}$ (see [5,6]). Specifically, we have

$$\mathbf{D}^T\mathbf{W} = -\mathbf{W}\mathbf{D}.$$

The medium matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_{\varepsilon} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\mu} \end{pmatrix},$$

and $\mathbf{M}_{\varepsilon}$ and $\mathbf{M}_{\mu}$ are diagonal and positive definite medium matrices with (averaged) permittivity and permeability values on the diagonal.

We write Eq. (1) in a more convenient form by premultiplying this equation by the inverse of the medium matrix. We obtain

$$\partial_t\mathbf{f} = \mathbf{A}\mathbf{f} - w(t)\mathbf{s}, \tag{2}$$

where $\mathbf{s} = \mathbf{M}^{-1}\mathbf{q}$ and where we have introduced the system matrix as $\mathbf{A} = -\mathbf{M}^{-1}\mathbf{D}$. The minus sign in the latter definition is for convenience only. The continuous-time solution of Eq. (2) is essentially given by the temporal convolution of the source wavelet $w(t)$ and the vector $\exp(\mathbf{A}t)\mathbf{s}$.

The system matrix $\mathbf{A}$ is skew-symmetric with respect to the diagonal and positive definite energy matrix $\mathbf{W}^{\text{en}} = \mathbf{W}\mathbf{M}$. More precisely, we have

$$\mathbf{A}^T\mathbf{W}^{\text{en}} = -\mathbf{W}^{\text{en}}\mathbf{A}$$

and consequently we know that there exists a matrix $\mathbf{V}$ such that

$$\mathbf{A}\mathbf{V} = \mathbf{V}\Lambda \quad \text{with} \quad \mathbf{V}^H\mathbf{W}^{\text{en}}\mathbf{V} = \mathbf{I}_n,$$

and $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ (see, for example [7]). Furthermore, all eigenvalues λ_i are pure imaginary and therefore we can also write $\Lambda = \mathrm{i}\Sigma$, where i is the imaginary unit, and $\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$ with $\sigma_j \in \mathbb{R}$, $j = 1, 2, \ldots, n$.

2.1. The FDTD update equations

From this moment on, we consider external electric-current type sources only ($\mathbf{k}^{\text{ext}} = \mathbf{0}$), since the analysis for external magnetic-type sources runs along similar lines.

Introduce the time instances $t_n = n\Delta t$, where $\Delta t > 0$ is the time step and n a nonnegative integer. The FDTD update equations are given by (see [3,4,8])

$$\mathbf{h}(t_{n+1/2}) = \mathbf{h}(t_{n-1/2}) - \Delta t\mathbf{M}_{\mu}^{-1}\mathbf{D}_{\text{e}}\mathbf{e}(t_n)$$

and

$$\mathbf{e}(t_{n+1}) = \mathbf{e}(t_n) - \Delta t\mathbf{M}_{\varepsilon}^{-1}\mathbf{D}_{\text{h}}\mathbf{h}(t_{n+1/2}) - \mathrm{I}_n\{w\}\mathbf{M}_{\varepsilon}^{-1}\mathbf{j}^{\text{ext}}$$

for $n = 0, 1, 2, \ldots$, with $\mathbf{h}(t_{-1/2}) = \mathbf{0}$, $\mathbf{e}(t_0) = \mathbf{0}$, and

$$\mathrm{I}_n\{w\} = \int_{\tau=t_n}^{t_{n+1}} w(\tau)\,\mathrm{d}\tau.$$

For simplicity, we take $w(t) = a\delta(t)$, where $\delta(t)$ is the Dirac delta function and a is a constant amplitude factor. The electromagnetic response to the desired source signature w (or any other signature for which the chosen spatial grid is suitable) can then be found by convolving $w(t)$ with the computed FDTD response. This approach is not necessary in what follows, but it simplifies the resulting formulas in the sense that now we do not have to drag along the source vector in the complete updating procedure. More precisely, with $w(t) = a\delta(t)$, we have

$$\mathrm{I}_0\{a\delta(t)\} = a \quad \text{and} \quad \mathrm{I}_n\{a\delta(t)\} = 0 \quad \text{for} \quad n \geqslant 1$$

and the update equations simplify to

$$\mathbf{h}(t_{n+1/2}) = \mathbf{h}(t_{n-1/2}) - \Delta t \mathbf{M}_\mu^{-1}\mathbf{D}_\mathrm{e}\mathbf{e}(t_n) \tag{3}$$

and

$$\mathbf{e}(t_{n+1}) = \mathbf{e}(t_n) - \Delta t \mathbf{M}_\varepsilon^{-1}\mathbf{D}_\mathrm{h}\mathbf{h}(t_{n+1/2}) \tag{4}$$

for $n \geqslant 1$ with

$$\mathbf{h}(t_{1/2}) = \mathbf{0} \quad \text{and} \quad \mathbf{e}(t_1) = -a\mathbf{M}_\varepsilon^{-1}\mathbf{j}^{\mathrm{ext}}.$$

3. Fibonacci polynomials and FDTD

To show the connection between FDTD and the Fibonacci polynomials, we first write the FDTD update Eqs. (3) and (4) in a more compact form. Introduce the FDTD field vectors

$$\mathbf{g}_n = \begin{pmatrix} \mathbf{0} \\ \mathbf{h}\left(t_{\frac{n+1}{2}}\right) \end{pmatrix} \quad \text{for} \quad n = 0, 2, 4, \ldots$$

and

$$\mathbf{g}_n = \begin{pmatrix} \mathbf{e}\left(t_{\frac{n+1}{2}}\right) \\ \mathbf{0} \end{pmatrix} \quad \text{for} \quad n = 1, 3, 5, \ldots$$

The update equations can now be written as

$$\mathbf{g}_{n+1} = \Delta t \mathbf{A}\mathbf{g}_n + \mathbf{g}_{n-1} \quad \text{for} \quad n = 1, 2, \ldots, \tag{5}$$

where $\mathbf{A}$ is the system matrix introduced in Section 2. Now the Fibonacci polynomials are defined by the recurrence formula (see, for example [9])

$$F_{n+1}(x) = xF_n(x) + F_{n-1}(x) \quad \text{for} \quad n = 1, 2, \ldots \tag{6}$$

with $F_0(x) = 0$ and $F_1(x) = 1$. The first few Fibonacci polynomials are

$$F_2(x) = x, \quad F_3(x) = x^2 + 1, \quad F_4(x) = x^3 + 2x, \quad F_5(x) = x^4 + 3x^2 + 1$$

and in general we have

$$F_n(x) = \frac{\left(\frac{x+\sqrt{x^2+4}}{2}\right)^n - \left(\frac{x-\sqrt{x^2+4}}{2}\right)^n}{\sqrt{x^2+4}},$$

for $n = 0, 1, \ldots$ Notice that the degree of $F_n(x)$ is $n - 1$ and the Fibonacci polynomials are normalized such that $F_n(1) = f_n$, where the f_n are the Fibonacci numbers. Finally, Fibonacci polynomials are related to Chebyshev polynomials of the second kind via the formula

$$F_{n+1}(2\mathrm{i}x) = \mathrm{i}^n U_n(x), \tag{7}$$

for $n = 0, 1, \ldots$

Comparing Eq. (5) with Eq. (6), we observe that

$$\mathbf{g}_n = F_n(\Delta t \mathbf{A})\mathbf{g}_1 \quad \text{for} \quad n = 1, 2, \ldots$$

In other words, the FDTD field vectors are Fibonacci polynomials in $\Delta t\mathbf{A}$ acting on the starting (source) vector $\mathbf{g}_1$. Having established this connection, it is now straightforward to show under what condition the FDTD method is stable. Specifically, using the spectral decomposition of the system matrix, we can write for the nth FDTD vector

$$\mathbf{g}_n = \mathbf{V}F_n(\Delta t\Lambda)\mathbf{V}^H\mathbf{W}^{\mathrm{en}}\mathbf{g}_1 = \mathbf{V}F_n\left(2\mathrm{i}\frac{\Delta t}{2}\Sigma\right)\mathbf{V}^H\mathbf{W}^{\mathrm{en}}\mathbf{g}_1 = \mathrm{i}^{n-1}\mathbf{V}U_{n-1}\left(\frac{\Delta t}{2}\Sigma\right)\mathbf{V}^H\mathbf{W}^{\mathrm{en}}\mathbf{g}_1,$$

where we have used Eq. (7). Since $U_n(x)$ is bounded on $(-1,1)$ for all $n \geqslant 0$ and since $U_n(x)$ becomes unbounded as n increases for any x not belonging to $(-1,1)$, we conclude from the above result that FDTD is stable if and only if

$$\Delta t < \frac{2}{\rho(\mathbf{A})},$$

where $\rho(\mathbf{A})$ is the spectral radius of matrix $\mathbf{A}$. This is the famous Courant-Friedrichs-Lewy (CFL) stability condition (see [5,8,10]). Explicit expressions for the spectral radius can only be given in special cases. For example, if uniform grids are applied to problems involving homogeneous media, then $\rho(\mathbf{A})$ can be determined explicitly (see, for example [6]). For general inhomogeneous media and grids with variable step sizes, we can either estimate the spectral radius by exploiting the specific structure of the system matrix (see [11]), or we can compute it numerically using an iterative eigensolver. For details, we refer to [11].

The connection between FDTD and Fibonacci polynomials can also be used to show how FDTD approximates the spectrum of the system matrix. To be specific, suppose that we have carried out k FDTD iterations using a time step Δt. We then have generated $2k$ FDTD vectors and we can summarize all these FDTD iterations into a single equation. In particular, setting $m = 2k$ and $\mathbf{G}_m = (\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_m)$, we have the summarizing equation

$$\mathbf{A}\mathbf{G}_m = \mathbf{G}_m\mathbf{T}_m + \frac{1}{\Delta t}\mathbf{g}_{m+1}\mathbf{e}_m^T, \tag{8}$$

where $\mathbf{e}_m$ is the mth column of the m-by-m identity matrix, and $\mathbf{T}_m$ is a tridiagonal and skew-symmetric matrix of order m given by

$$\mathbf{T}_m = \frac{1}{\Delta t}\begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & -1 & & \\ & & \ddots & -1 \\ & & 1 & 0 \end{pmatrix}.$$

The eigenproblem for matrix $\mathbf{T}_m$ can be solved explicitly. In particular, the eigenvalues of matrix $\mathbf{T}_m$ are given by

$$\zeta_k = 2\mathrm{i}\frac{z_k}{\Delta t} \quad \text{for} \quad k = 1, 2, \ldots, m, \tag{9}$$

where

$$z_k = \cos\left(\frac{k\pi}{m+1}\right) \quad \text{for} \quad k = 1, 2, \ldots, m$$

are the roots of the Chebyshev polynomial U_m (the roots of the Fibonacci polynomial F_{m+1} are $2\mathrm{i}z_k$, cf. Eq. (7)). We observe that the only way to adapt FDTD to the spectrum of the system matrix is through the time step Δt. By selecting the time step as large as possible, that is, as close to the CFL upper bound as possible, the FDTD eigenvalues are essentially distributed over the spectral interval $\mathrm{i}\rho(\mathbf{A})(-1,1)$ of the system matrix. More precisely, with

$$\Delta t = (1-\epsilon)\frac{2}{\rho(\mathbf{A})}, \quad \text{and} \quad \epsilon > 0,$$

the FDTD eigenvalues are given by

$$\zeta_k = \mathrm{i}\rho(\mathbf{A})z_k[1 + \mathrm{o}(1)] \quad \text{as } \epsilon \downarrow 0.$$

To put it differently, by selecting the time step as large as possible, we take care of the extremal eigenvalues only. There is no other spectral adaptation in FDTD. This is to be contrasted with approximation methods based on the Lanczos algorithm for skew-symmetric matrices such as the Spectral Lanczos Decomposition Method (SLDM, see, for example [12,13]). The Lanczos algorithm also computes a decomposition of a form as given by Eq. (8), but the tridiagonal Lanczos matrix automatically adapts itself to the spectrum of the system matrix as can be seen from the coefficients located along its upper and lower diagonal. In general, these coefficients are not constant (as in FDTD) and change at every iteration. Loosely speaking, the reciprocals of the Lanczos recurrence coefficients act as time steps in a Lanczos method and are generated automatically. There is no explicit time step selection that fixes all eigenvalues as in FDTD.

4. Conclusions

In this paper, we have shown that FDTD field approximations are Fibonacci polynomials in $\Delta t\mathbf{A}$, where Δt is the time step and $\mathbf{A}$ the first-order Maxwell system matrix. We have also shown that the CFL stability condition is easily obtained by exploiting the connection between Fibonacci polynomials and Chebyshev polynomials of the second kind. Furthermore, we have shown that there is no automatic spectral adaptation in FDTD. We can only match FDTD to the spectral width of the system matrix by selecting the time step as close to the CFL upper bound as possible. Finally, we like to point out that it may be possible to apply (a modification of) the present analysis to other FDTD like schemes such as the ones presented in [14,15].

Acknowledgment

The author thanks the reviewers for providing him with a number of important references.

References

[1] K.S. Yee, Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media, IEEE Trans. Antennas Propag. 14 (1966) 302–307.
[2] A. Taflove, Application of the Finite-Difference Time-Domain method to sinusoidal steady-state electromagnetic penetration problems, IEEE Trans. Electromagn. Compat. 22 (1982) 191–202.
[3] A. Taflove, S.C. Hagness, Computational Electrodynamics. The Finite-Difference Time-Domain Method, third ed., Artech House, Boston, 2005.
[4] D.B. Davidson, Computational Electromagnetics for RF and Microwave Engineering, Cambridge University Press, Cambridge, 2005.
[5] B. Denecker, L. Knockaert, F. Olyslager, D. De Zutter, A new state-space-based algorithm to assess the stability of the Finite-Difference Time-Domain method for 3D finite inhomogeneous problems, AEU-Int. J. Electron. Commun. 58 (2004) 339–348.
[6] R.F. Remis, Global form of the 2d finite-difference Maxwell system and its symmetry properties, in: Proceedings of 2002 IEEE Antennas and Propagation Symposium, San Antonio, Texas, 2002, pp. 624–627.
[7] C.D. Meyer, Matrix Analysis and Applied Linear Algebra, Society for Industrial and Applied Mathematics, Philadelphia, 2000.
[8] R.F. Remis, On the stability of the Finite-Difference Time-Domain method, J. Comput. Phys. 163 (2000) 249–261.
[9] T. Koshy, Fibonacci and Lucas Numbers With Applications, Wiley-Interscience, New York, 2001.
[10] A. Taflove, M.E. Brodwin, Numerical solution of steady-state electromagnetic scattering problems using the time-dependent Maxwell's equations, IEEE Trans. Microwave Theory Tech. 23 (1975) 623–630.
[11] R.F. Remis, Stability of FDTD on nonuniform grids for Maxwell's equations in lossless media, J. Comput. Phys. 218 (2006) 594–606.
[12] V.L. Druskin, L.A. Knizhnerman, Two polynomial methods for calculating functions of symmetric matrices, U.S.S.R. Comput. Math. Math. Phys. 29 (1989) 112–121.
[13] R.F. Remis, P.M. van den Berg, A modified Lanczos algorithm for the computation of transient electromagnetic wavefields, IEEE Trans. Microwave Theory Tech. 45 (December) (1997) 2139–2149.
[14] T. Rylander, A. Bondeson, Stable FEM-FDTD hybrid method for Maxwell's equations, Comput. Phys. Commun. 125 (2000) 75–82.
[15] N. Marais, D.B. Davidson, Efficient high-order time domain hybrid implicit/explicit FEM methods for microwave electromagnetics, Electromagnetics 30 (2010) 127–148.