HIGH RATE SPHERICAL QUANTIZATION OF SINUSOIDAL PARAMETERS

Pim Korten, Jesper Jensen and Richard Heusdens

Information and Communication Theory Group, Delft University of Technology
Mekelweg 4, 2628 CD Delft, The Netherlands
phone:+31 (0)15 27 82188, fax:+31 (0)15 27 81843
email: {p.e.l.korten, j.jensen, r.heusdens}@ewi.tudelft.nl

ABSTRACT

Quantization of sinusoidal model parameters is of importance in e.g. low-rate audio coding. In this work we introduce entropy constrained unrestricted spherical quantization, where amplitude, phase and frequency are quantized dependently. We derive a high-rate approximation of the average ℓ_2-distortion and use this to analytically derive formulas for optimal spherical scalar quantizers. These quantizers minimize the average distortion, while the corresponding quantization indices satisfy an entropy constraint. The quantizers turn out to be ¤exible and of low complexity, in the sense that they can be determined for varying entropy constraints without any iterative retraining procedures. As a consequence of minimizing the ℓ_2-norm of the (quantization) error signal, the quantizers depend on both the shape and length of the analysis/synthesis window.

1. INTRODUCTION

Parametric coding has proved to be very effective for representing audio signals at low bit rates [1, 2, 3]. A typical parametric coder uses a decomposition of an audio signal into three components: a sinusoidal component, a noise component and a transient component, which are coded by separate subcoders. The sinusoidal component, represented by the parameters amplitude, phase and frequency, is perceptually the most important of the three, and in typical low-rate audio coders the main part of the bit budget is used for this component [3]. Often, the bit budget available for encoding sinusoids is given a priori, e.g. by a rate-distortion control algorithm which distributes the total bit rate over the subcoders. For this reason it is desirable to have simple and ¤exible quantizers which can adapt to changing bit rate requirements without any sort of retraining or iterations. Finding ef£cient quantizers for the sinusoidal component and its corresponding parameters is therefore critical.

In [4], entropy constrained unrestricted polar quantization (ECUPQ) is introduced, in which only amplitude and phase parameters are quantized. The term unrestricted refers to the fact that amplitude and phase are quantized dependently, that is, phase quantization depends on the input amplitude. The derivations in the cited paper are done under a high rate assumption, i.e. a very large number of quantization cells. Furthermore this assumption also implies that the probability density functions of the input variables are approximately constant in each quantization cell. The resulting quantizers turn out to be ¤exible and of low complexity. A shortcoming of this work, however, is that it does not consider quantization of frequency parameters.

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In this paper ECUPQ will be generalized to include frequency quantization. We denote this extended scheme by entropy constrained unrestricted spherical quantization (ECUSQ). Analogously with ECUPQ, amplitude, phase and frequency are quantized dependently. Using high-rate assumptions, we derive optimal amplitude, phase and frequency quantizers, which minimize the distortion, while satisfying an entropy constraint. Furthermore the rate distribution between amplitude, phase and frequency will be discussed. Note that since we also consider frequency quantization and hence consider multiple samples of a sinusoid, the distortion measure we use will be dependent on the frame length and analysis/synthesis window.

The remainder of this paper is organized as follows. In Section 2.1 we will derive a high-rate expression for the average distortion for a single sinusoid. In Section 2.2 we minimize this expression under an entropy constraint, resulting in the optimal quantizers and a distortion-rate relation. The multiple sinusoid case will be considered in Section 2.3. In Section 3, the found theoretical distortion-rate curve will be compared to a practically obtained curve, and the distribution of the rate between amplitude phase and frequency will be discussed. Finally, some conclusive remarks are given in Section 4.

2. ENTROPY CONSTRAINED UNRESTRICTED SPHERICAL QUANTIZATION

2.1 High-rate expression for the average distortion - single sinusoid

In this section we will derive a high-rate approximation for the average distortion concerning a single sinusoid. Let the original and quantized spherical representation of a complex sinusoid be denoted by $ae^{j(\nu n+\phi)}$ and $\tilde{a}e^{j(\tilde{\nu}n+\tilde{\phi})}$ respectively, for $n = n_0, \ldots, n_0+N-1$, where a is amplitude, ϕ is phase, ν is frequency, $n_0 \in \mathbb{Z}$, and N is the frame length. Furthermore, let $\varepsilon(n)$ denote the difference between the original and quantized sinusoid, and let w be the window de£ning the signal segment. The average distortion corresponding to the ℓ_2-distortion measure is then given by

$$D = E\left(d(a,\phi,\nu,\tilde{a},\tilde{\phi},\tilde{\nu})\right), \tag{1}$$

where $E(\cdot)$ denotes expectation, and

$$\begin{aligned}
&d(a,\phi,\nu,\tilde{a},\tilde{\phi},\tilde{\nu}) \\
&=\sum_{n=n_0}^{n_0+N-1} |w(n)\varepsilon(n)|^2 \\
&=\sum_{n=n_0}^{n_0+N-1} \left|w(n)ae^{j(\nu n+\phi)}-w(n)\tilde{a}e^{j(\tilde{\nu}n+\tilde{\phi})}\right|^2 \\
&=\|w\|^2(a^2+\tilde{a}^2)-2a\tilde{a}\sum_{n=n_0}^{n_0+N-1} w(n)^2\cos\left((\nu-\tilde{\nu})n+\phi-\tilde{\phi}\right)
\end{aligned} \tag{2}$$

denotes the ℓ_2-error, and $\|w\|^2=\sum_{n=n_0}^{n_0+N-1} w(n)^2$ the 2-norm of the window w. To derive a high-rate approximation of the average distortion (1), we first determine the ℓ_2-distortion in a quantization cell, which can be found by averaging over the corresponding amplitude, phase and frequency quantization intervals X_a, X_ϕ and X_ν with lengths $\triangle a$, $\triangle\phi$ and $\triangle\nu$, respectively:

$$\begin{aligned}
&\bar{d}(\tilde{a},\tilde{\phi},\tilde{\nu},\triangle a,\triangle\phi,\triangle\nu) \\
&=\frac{\int_{X_a}\int_{X_\phi}\int_{X_\nu} f_{A,\Phi,F}(a,\phi,\nu)d(a,\phi,\nu,\tilde{a},\tilde{\phi},\tilde{\nu})d\nu d\phi da}{\int_{X_a}\int_{X_\phi}\int_{X_\nu} f_{A,\Phi,F}(a,\phi,\nu)d\nu d\phi da}.
\end{aligned} \tag{3}$$

Under high-rate assumptions, the joint probability density function $f_{A,\Phi,F}(a,\phi,\nu)$ is approximately constant over a quantization cell. Consequently, the quantization points are located in the center of the quantization intervals. Using these assumptions in (3) and approximating the sines with their Taylor expansions, we finally obtain

$$\bar{d}(\tilde{a},\triangle a,\triangle\phi,\triangle\nu)\approx\frac{\|w\|^2}{12}\left(\triangle a^2+\tilde{a}^2\left(\triangle\phi^2+\sigma^2\triangle\nu^2\right)\right) \tag{4}$$

where $\sigma^2=\frac{1}{\|w\|^2}\sum_{n=n_0}^{n_0+N-1} w(n)^2n^2$.

A high-rate approximation for (1) can now be found by averaging the distortion (4) over all quantization cells. Let the amplitude, phase and frequency quantization indices corresponding to a quantization cell be denoted as i_a, i_ϕ and i_ν, respectively, and let I_a, I_ϕ and I_ν denote their corresponding alphabets. We obtain

$$\begin{aligned}
D=&\sum_{i_a\in I_a}\sum_{i_\phi\in I_\phi}\sum_{i_\nu\in I_\nu} p_{I_a,I_\phi,I_\nu}(i_a,i_\phi,i_\nu)\bar{d}(\tilde{a},\triangle a,\triangle\phi,\triangle\nu)_{i_a,i_\phi,i_\nu} \\
\approx&\frac{\|w\|^2}{12}\iiint f_{A,\Phi,F}(a,\phi,\nu)\Big(g_A^{-2}(a,\phi,\nu) \\
&+a^2\left(g_\Phi^{-2}(a,\phi,\nu)+\sigma^2 g_F^{-2}(a,\phi,\nu)\right)\Big)d\nu d\phi da
\end{aligned} \tag{5}$$

where $p_{I_a,I_\phi,I_\nu}(i_a,i_\phi,i_\nu)$ is the probability of the cell corresponding to the quantization indices i_a, i_ϕ and i_ν. In this derivation we used high-rate assumptions and hence substituted sums by integrals and quantization step sizes by so-called quantization point density functions [5, 6], which when integrated over a region S gives the total number of quantization levels within S. In the case of one-dimensional quantizers, this means that the quantizer step sizes are just given by the reciprocal values of the point densities, that is, $g=\Delta^{-1}$. In high-rate theory, quantizers are described by these density functions, without exactly specifying the location of the quantization points. Note that since we consider unrestricted quantization, the quantization point density functions depend on all three parameters.

2.2 Entropy-constrained minimization of the average distortion - single sinusoid

In this section we will determine the quantization point density functions that minimize the average distortion (5), while satisfying the entropy constraint $H(I_a,I_\phi,I_\nu)=H_t$, where H_t is the given total target entropy, and $H(I_a,I_\phi,I_\nu)$ is the joint entropy of amplitude, phase and frequency quantization indices. The joint entropy $H(I_a,I_\phi,I_\nu)$ can be approximated, under high-rate assumptions, by

$$\begin{aligned}
H(I_a,I_\phi,I_\nu)\approx& h(A,\Phi,F) \\
&+\iiint f_{A,\Phi,F}(a,\phi,\nu)\log_2(g_A(a,\phi,\nu))d\nu d\phi da \\
&+\iiint f_{A,\Phi,F}(a,\phi,\nu)\log_2(g_\Phi(a,\phi,\nu))d\nu d\phi da \\
&+\iiint f_{A,\Phi,F}(a,\phi,\nu)\log_2(g_F(a,\phi,\nu))d\nu d\phi da,
\end{aligned}$$

where $h(A,\Phi,F)$ is the joint differential entropy of amplitude, phase and frequency, which is independent of the quantization point density functions. Using this approximation, we rewrite the entropy constraint as $\tilde{H}(I_a,I_\phi,I_\nu)=\tilde{H}_t$, where we subtracted $h(A,\Phi,F)$ from both sides of the original constraint equality. We now have a constrained minimization problem that can be solved using the method of Lagrange multipliers, turning it into an unconstrained problem. The criterion to minimize then is

$$\begin{aligned}
\eta=&D+\lambda\Big(\iiint f_{A,\Phi,F}(a,\phi,\nu)\log_2(g_A(a,\phi,\nu))d\nu d\phi da \\
&+\iiint f_{A,\Phi,F}(a,\phi,\nu)\log_2(g_\Phi(a,\phi,\nu))d\nu d\phi da \\
&+\iiint f_{A,\Phi,F}(a,\phi,\nu)\log_2(g_F(a,\phi,\nu))d\nu d\phi da\Big),
\end{aligned}$$

where λ is the Lagrange multiplier, and D is given by (5). Evaluating the Euler-Lagrange equations with respect to $g_A(a,\phi,\nu)$, $g_\Phi(a,\phi,\nu)$ and $g_F(a,\phi,\nu)$ individually, we obtain

$$g_A(a,\phi,\nu)=g_A=\left(\frac{\|w\|^2}{6\lambda\log_2(e)}\right)^{\frac{1}{2}}, \tag{6}$$

$$g_\Phi(a,\phi,\nu)=g_\Phi(a)=\left(\frac{\|w\|^2a^2}{6\lambda\log_2(e)}\right)^{\frac{1}{2}}, \tag{7}$$

$$g_F(a,\phi,\nu)=g_F(a)=\left(\frac{\sigma^2\|w\|^2a^2}{6\lambda\log_2(e)}\right)^{\frac{1}{2}}. \tag{8}$$

Substituting these three expressions into the entropy constraint, we find the optimal value of the Lagrange multiplier:

$$\lambda=\frac{\|w\|^2 2^{-\frac{2}{3}\left(\tilde{H}_t-2b(A)-\log_2(\sigma)\right)}}{6\log_2(e)},$$

where $\tilde{H}_t=H_t-h(A,\Phi,F)$ and $b(A)=\int f_A(a)\log_2(a)da$ are introduced for notational simplicity. Substituting this result back in (6), (7) and (8), we find the optimal high-rate

ECUSQ quantizers for the case of a single sinusoid and the ℓ_2-distortion measure:

$$g_A = 2^{\frac{1}{3}\left(\tilde{H}_t - 2b(A) - \log_2(\sigma)\right)}, \quad (9)$$

$$g_\Phi(a) = a 2^{\frac{1}{3}\left(\tilde{H}_t - 2b(A) - \log_2(\sigma)\right)}, \quad (10)$$

$$g_F(a) = \sigma a 2^{\frac{1}{3}\left(\tilde{H}_t - 2b(A) - \log_2(\sigma)\right)}. \quad (11)$$

We see that the optimal amplitude quantizer is uniform, and both the optimal phase and frequency quantizer are uniform in phase and frequency and depend linearly on amplitude. Furthermore, unlike the quantizers derived in [4], the ECUSQ quantizers in (9)-(11) depend on the signal frame length N and the analysis/synthesis window w (through σ).

The minimal average distortion for ECUSQ can now be found by substituting (9), (10) and (11) in (5):

$$D_{ECUSQ} = \frac{\|w\|^2 2^{-\frac{2}{3}\left(\tilde{H}_t - 2b(A) - \log_2(\sigma)\right)}}{4}. \quad (12)$$

It is not dif£cult to show that if w is an evenly-symmetric window, the distortion (12) is minimal for $n_0 = -\frac{(N-1)}{2}$. We then have $\sigma^2 = \frac{1}{12}(N^2 - 1)$. We assume this to be the case in the remainder of this work.

2.3 Multiple sinusoids

In the case of L independent sinusoids, the total average distortion is determined by $D_{tot} = \frac{1}{L}\sum_{l=1}^{L} D_l = D$. Since the expression for the distortion of a single sinusoid, as de£ned in (1), is a squared-error distortion measure, each sinusoid gives the same contribution to the total distortion. The entropy constraint is given by $\frac{1}{L}\sum_{l=1}^{L} H_l(I_a, I_\phi, I_\nu) = H_t$, which simpli£es to $H(I_a, I_\phi, I_\nu) = H_t$, since each sinusoid also gives the same contribution to the total entropy of quantization indices. We see that we end up with exactly the same constrained optimization problem as for a single sinusoid, which means that the quantizers (9), (10) and (11) are also optimal for multiple sinusoids for this distortion measure. In [4] a weighted distortion measure is used, such that each sinusoid is weighted differently, depending on its perceptual importance. It is straight-forward to make this extension here as well; in this case the optimal quantizers will depend on the weights of the sinusoids.

3. EXPERIMENTAL RESULTS

In this section the theoretical rate-distortion function derived in (12) for ECUSQ will be compared to a practically obtained rate-distortion curve. Secondly, the distribution of bits between amplitude, phase and frequency in the optimal ECUSQ quantizer, and its dependency on the frame length will be discussed.

Let X, Y and Z denote three independent Gaussian variables, with zero mean and unit variance. The corresponding spherical variables amplitude, phase and frequency are then de£ned by respectively

$$A = \sqrt{X^2 + Y^2 + Z^2},$$
$$\Phi = \arctan\left(\frac{Y}{X}\right),$$
$$F = \arctan\left(\frac{Z}{\sqrt{X^2 + Y^2}}\right).$$

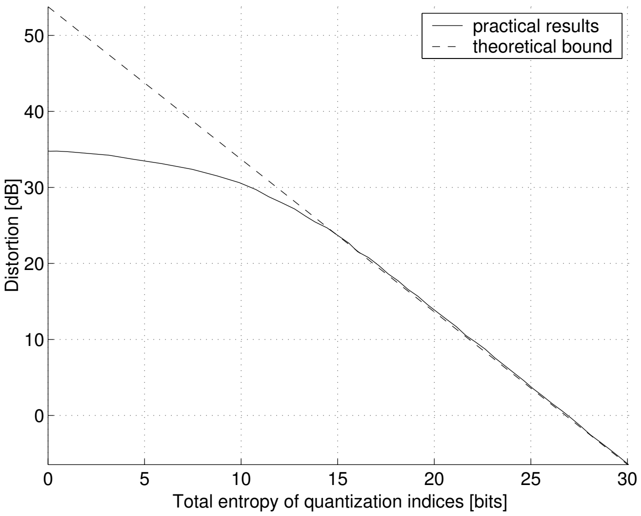

Figure 1: Theoretical versus practical distortion-rate performance for $N = 1024$.

Using the rules for computing probability density functions of a transformation of random variables, it can be shown that the amplitude A has the Maxwell density $M(1)$, the phase Φ has the uniform density $U(0, 2\pi)$ and the frequency F has a probability density function given by $f_F(\nu) = \frac{sin(\nu)}{2}$ for $0 \leq \nu \leq \pi$. It can be veri£ed that A, Φ and F are independent.

Using these distributions, a large number, M, of triplets $\{a, \phi, \nu\}$ are generated, and subsequently quantized with the quantizers derived in (9), (10) and (11) for a given target entropy. Using (2), the quantization distortion for each triplet is determined, and averaged over all triplets. Computing the entropy of the M quantized triplets then gives us a rate distortion pair. Repeating this procedure for several different target entropies H_t, we obtain a practical rate distortion curve as plotted in £gure 1, where we used $M = 10000$. In the same £gure the theoretical rate distortion curve given by (12) is plotted, where we used a rectangular window with length $N = 1024$. It can clearly be seen that the curves converge towards each other, which veri£es that the expression (12) for the average distortion is indeed a good approximation at high rates. At an entropy of 30 bits the difference between the curves is only 0.1 dB, and for higher rates this difference decreases. For low rates it is clear that the approximation (12) is not valid anymore.

The distribution of the rate between amplitude, phase and frequency in the optimal ECUSQ quantizer can be found by determining the entropies of the quantization indices $H(I_a)$, $H(I_\phi|I_a)$ and $H(I_\nu|I_a)$. Using high-rate assumptions we obtain

$$H(I_a) = -\sum_{i_a \in I_a} p_{I_a}(i_a) \log_2(p_{I_a}(i_a))$$
$$\approx h(A) + \log_2(g_A),$$
$$H(I_\phi|I_a) = -\sum_{i_a \in I_a}\sum_{i_\phi \in I_\phi} p_{I_a, I_\phi}(i_a, i_\phi) \log_2\left(p_{I_\phi|I_a}(i_\phi|i_a)\right)$$
$$\approx h(\Phi|A) + \int f_A(a) \log_2(g_\Phi(a)) da,$$

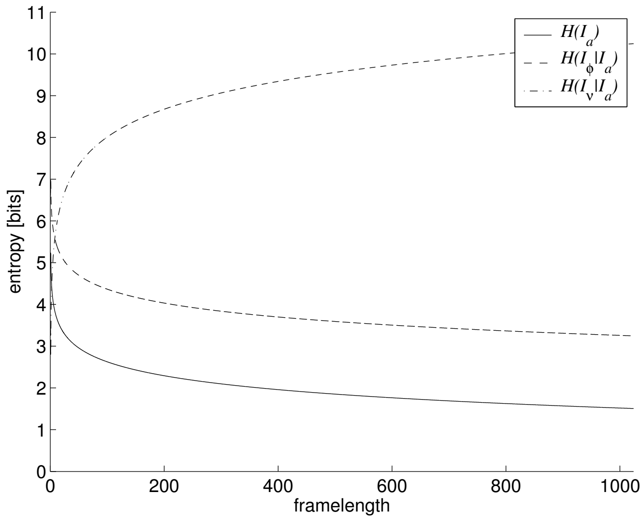

Figure 2: Entropies of quantization indices as a function of frame length for $H_t = 15$.

and in the same way

$$H(I_\nu|I_a) \approx h(F|A) + \int f_A(a)\log_2(g_F(a))da,$$

where $h(A)$, $h(\Phi|A)$ and $h(F|A)$ are differential entropies. Substituting the optimal quantizers (9), (10) and (11) into these equations, and assuming the same distributions as earlier in this section (so A, Φ and F are independent) we £nally obtain

$$H(I_a) \approx \frac{1}{3}(H - \log_2(\sigma) - 2.27),$$
$$H(I_\phi|I_a) \approx \frac{1}{3}(H - \log_2(\sigma) + 2.95),$$
$$H(I_\nu|I_a) \approx \frac{1}{3}(H + 2\log_2(\sigma) - 0.68).$$

Here we used that $h(A) = 1.437$, $h(\Phi|A) = h(\Phi) = 2.651$, $h(F|A) = h(F) = 1.443$ and $b(A) = 0.526$. For a £xed target rate H_t, these entropies only depend on the frame length N. In Figure 2 the entropies of the quantization indices are plotted as a function of N for $H_t = 15$. We see that phase will always be assigned 1.74 bits more than amplitude. Furthermore, if the frame length N is increased, more bits will be assigned to frequency, and hence less to amplitude and phase. This can be expected since for increasing frame length, the frequency quantization error grows more rapidly than the amplitude and phase quantization error. Consequently, more bits will have to be assigned to the frequency quantizer in order to keep the distortion minimal. Such a frame length dependent quantization is important in coding schemes where variable segment length analysis is used, see e.g. [7, 8].

4. CONCLUSIVE REMARKS

In this work we analytically derived optimal entropy-constrained unrestricted spherical quantizers, for quantization of amplitude, phase and frequency parameters. These derivations were done under a high-rate assumption which increases the simplicity of the derivations signi£cantly. The quantizers turned out to be ¤exible and of low complexity in the sense that they can adapt easily to changing bit rate requirements without any retraining or iterative procedures. As a consequence of minimizing the ℓ_2-norm of the (quantization) error signal, the quantizers depend on both the shape and length of the analysis/synthesis window.

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