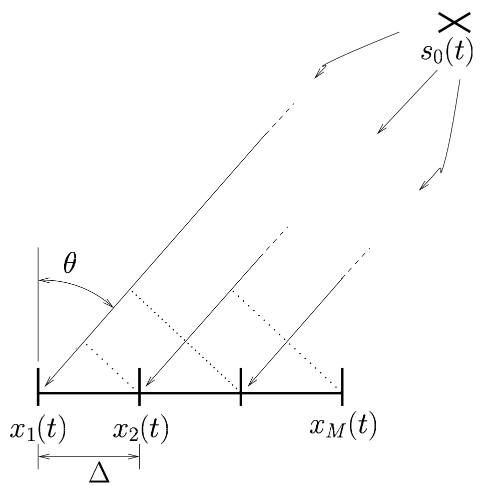
# 5. DIRECTION ESTIMATION USING ESPRIT

## Problem

From the output of a **uniform linear antenna array**, estimate the angles of arrival




$$\mathbf{X} = \mathbf{ABS} = \mathbf{a}(\theta_1)\beta_1\mathbf{s}_1 + \mathbf{a}(\theta_2)\beta_2\mathbf{s}_2 + \cdots$$

- General parametric procedure for DOA estimation:

$$\arg\min_{\{\theta_i\}} \|\mathbf{X} - \mathbf{ABS}\|_F^2, \qquad \text{where} \quad \mathbf{A} = [\mathbf{a}(\theta_1)\ \ \mathbf{a}(\theta_2)\cdots]$$

- We have to know the array manifold $\mathbf{a}(\theta)$.

# The ESPRIT algorithm

- For a uniform linear array, $\phi = e^{j\Delta 2\pi \sin(\theta)}$, and thus

$$\mathbf{a}(\theta) = \begin{bmatrix} 1 \\ \phi \\ \phi^2 \\ \vdots \\ \phi^{M-1} \end{bmatrix} \quad \begin{matrix} \left.\begin{matrix} 1 \\ \vdots \\ \phi^{M-2} \end{matrix}\right\} \mathbf{a}_x(\theta) \\ \left.\begin{matrix} \phi \\ \vdots \\ \phi^{M-1} \end{matrix}\right\} \mathbf{a}_y(\theta) \end{matrix}$$

- Shift-invariance property:

$$\mathbf{a}_x(\theta) := \begin{bmatrix} 1 \\ \phi \\ \vdots \\ \phi^{M-2} \end{bmatrix}, \qquad \mathbf{a}_y(\theta) := \begin{bmatrix} \phi \\ \phi^2 \\ \vdots \\ \phi^{M-1} \end{bmatrix}, \qquad \text{so that} \quad \mathbf{a}_y(\theta) = \mathbf{a}_x(\theta)\phi$$

# The ESPRIT algorithm

- Let us group the first and last $M-1$ antennas

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_{M-1}(t) \end{bmatrix}, \qquad \mathbf{y}(t) = \begin{bmatrix} x_2(t) \\ \vdots \\ x_M(t) \end{bmatrix}, \qquad \begin{array}{rcl} \mathbf{X} & = & [\mathbf{x}(0) \quad \cdots \quad \mathbf{x}(N-1)] \\ \mathbf{Y} & = & [\mathbf{y}(0) \quad \cdots \quad \mathbf{y}(N-1)] \end{array}$$

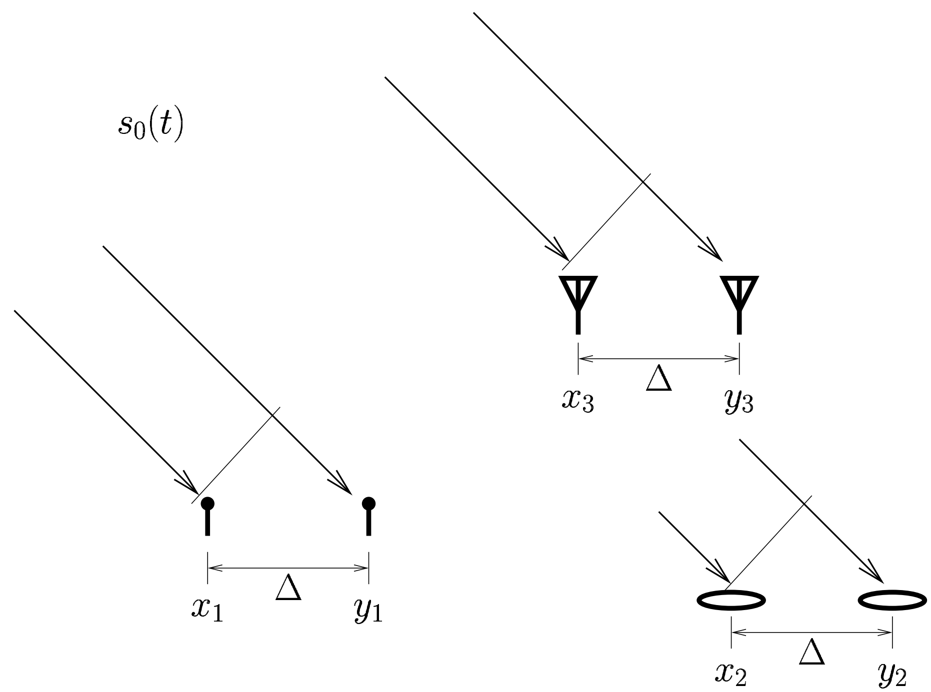
- From the shift-invariance property:

$$\begin{array}{rclcrcl} \mathbf{x}(t) & = & \sum_{k=1}^{d} \mathbf{a}_x(\theta_k)\beta_k s_k(t) & \Rightarrow & \mathbf{X} & = & \mathbf{ABS} \\ \mathbf{y}(t) & = & \sum_{k=1}^{d} \mathbf{a}_y(\theta_k)\beta_k s_k(t) = \sum_{k=1}^{d} \mathbf{a}_x(\theta_k)\phi_k\beta_k s_k(t) & \Rightarrow & \mathbf{Y} & = & \mathbf{A\Theta BS} \end{array}$$

where

$$\mathbf{A} = [\mathbf{a}_x(\theta_1) \quad \cdots \quad \mathbf{a}_x(\theta_d)], \qquad \mathbf{\Theta} = \begin{bmatrix} \phi_1 & & \\ & \ddots & \\ & & \phi_d \end{bmatrix}, \qquad \phi_k = e^{j2\pi\Delta\sin(\theta_k)}$$

## More general “doublet” antenna structure




$$x_i(t) = \sum_{k=1}^{d} a_{i,k} \beta_k s_k(t) \,, \qquad y_i(t) = \sum_{k=1}^{d} a_{i,k} e^{j2\pi\Delta \sin(\theta_k)} \beta_k s_k(t)$$

$$\mathbf{z}(t) = \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{y}(t) \end{bmatrix} \qquad \Rightarrow \qquad \mathbf{Z} = \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix} = \begin{bmatrix} \mathbf{A} \\ \mathbf{A}\mathbf{\Theta} \end{bmatrix} \mathbf{BS}$$

# The ESPRIT algorithm

Given the data matrix $\mathbf{Z}$ from all antennas

$$\mathbf{Z} = \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix} = \mathbf{A}_z \mathbf{B} \mathbf{S}, \qquad \mathbf{A}_z = \begin{bmatrix} \mathbf{A} \\ \mathbf{A}\Theta \end{bmatrix}$$

- Note that $\mathbf{Z}$ has rank $d$. Compute an SVD of $\mathbf{Z}$:

$$\mathbf{Z} = \hat{\mathbf{U}}_z \hat{\Sigma}_z \hat{\mathbf{V}}_z^{\mathrm{H}}$$

$\hat{\mathbf{U}}_z : 2M \times d$ has $d$ columns which together span the column space of $\mathbf{Z}$

- Note that $\hat{\mathbf{U}}_z$ spans the same space as $\mathbf{A}_z$, hence there exists a $d \times d$ matrix $\mathbf{T}$:

$$\hat{\mathbf{U}}_z = \mathbf{A}_z \mathbf{T} = \begin{bmatrix} \mathbf{A}\mathbf{T} \\ \mathbf{A}\Theta\mathbf{T} \end{bmatrix}$$

- Split $\mathbf{Z}$ into $\mathbf{X}$ and $\mathbf{Y}$, and $\hat{\mathbf{U}}_z$ accordingly into $\hat{\mathbf{U}}_x$ and $\hat{\mathbf{U}}_y$, then

$$\begin{cases} \hat{\mathbf{U}}_x = \mathbf{A}\mathbf{T} \\ \hat{\mathbf{U}}_y = \mathbf{A}\Theta\mathbf{T} \end{cases}$$

# The ESPRIT algorithm

$$\begin{cases} \hat{\mathbf{U}}_x &= \mathbf{AT} \\ \hat{\mathbf{U}}_y &= \mathbf{A\Theta T} \end{cases}$$

- Note that $\hat{\mathbf{U}}_x^\dagger = (\mathbf{T}^{\mathrm{H}}\mathbf{A}^{\mathrm{H}}\mathbf{AT})^{-1}\mathbf{T}^{\mathrm{H}}\mathbf{A}^{\mathrm{H}} = \mathbf{T}^{-1}\mathbf{A}^\dagger$ so that

$$\hat{\mathbf{U}}_x^\dagger \hat{\mathbf{U}}_y = \mathbf{T}^{-1}\boldsymbol{\Theta}\mathbf{T}\,.$$

- Thus, $\mathbf{T}^{-1}$ and $\boldsymbol{\Theta}$ are given by the eigenvectors and eigenvalues of $\hat{\mathbf{U}}_x^\dagger \hat{\mathbf{U}}_y$.
- From $\boldsymbol{\Theta}$ we can derive $\{\phi_k\}$ and hence $\{\theta_k\}$
- From $\mathbf{T}$ we can derive a zero-forcing beamformer on $\mathbf{Z}$ as

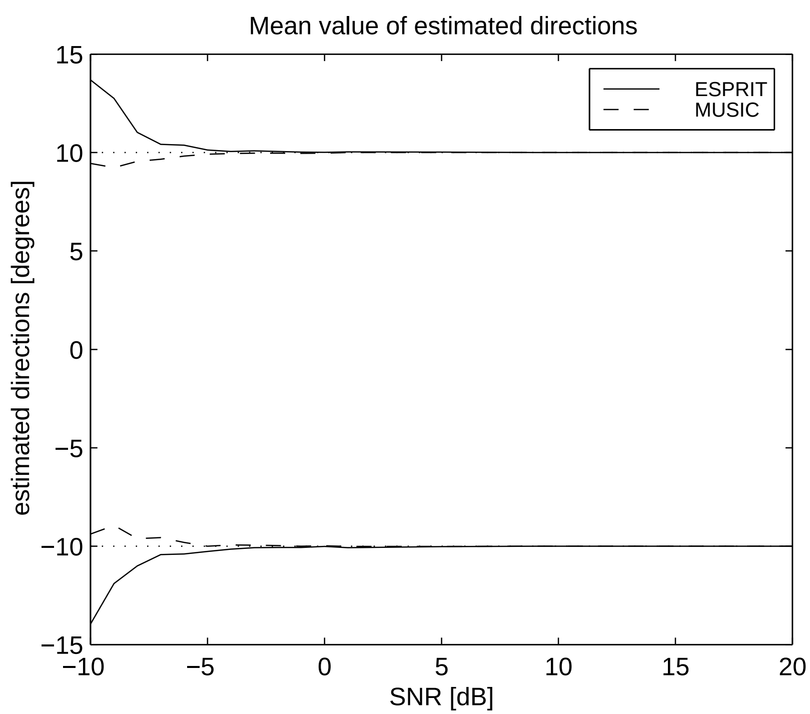
$$\mathbf{W} = \hat{\mathbf{U}}_z \mathbf{T}^{\mathrm{H}}$$

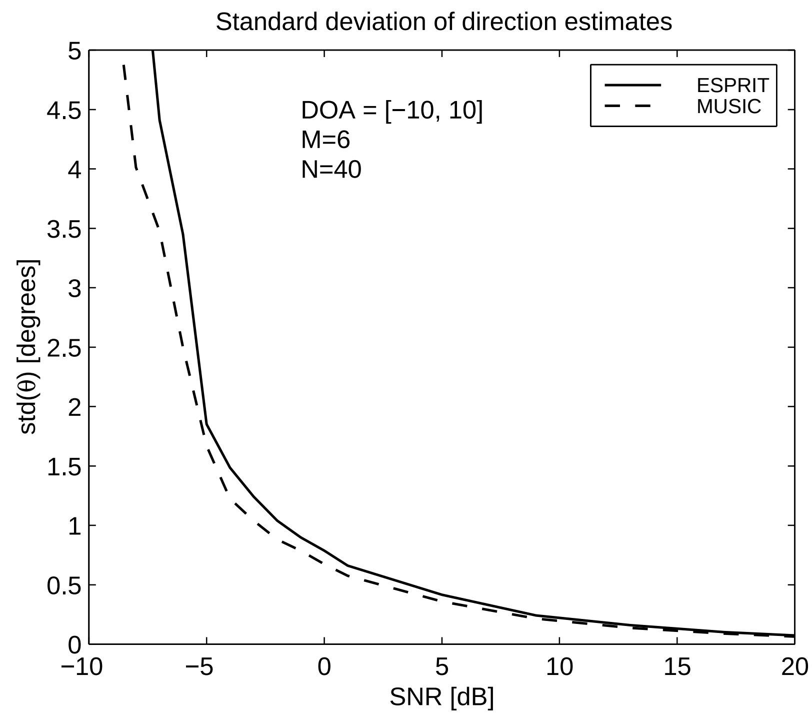
Proof:

$$\begin{aligned} \mathbf{Z} &= \hat{\mathbf{U}}_z \cdot \hat{\Sigma}_z \hat{\mathbf{V}}_z^{\mathrm{H}} = \mathbf{A}_z\mathbf{S} = \mathbf{A}_z\mathbf{T}\cdot\mathbf{T}^{-1}\mathbf{S} \\ \Rightarrow \quad \mathbf{T}^{-1}\mathbf{S} &= \hat{\Sigma}_z \hat{\mathbf{V}}_z^{\mathrm{H}} = \hat{\mathbf{U}}_z^{\mathrm{H}}\mathbf{Z} \\ \Rightarrow \quad \mathbf{S} &= \mathbf{T}\hat{\mathbf{U}}_z^{\mathrm{H}}\mathbf{Z} \end{aligned}$$

# The ESPRIT algorithm

## Performance (varying SNR)



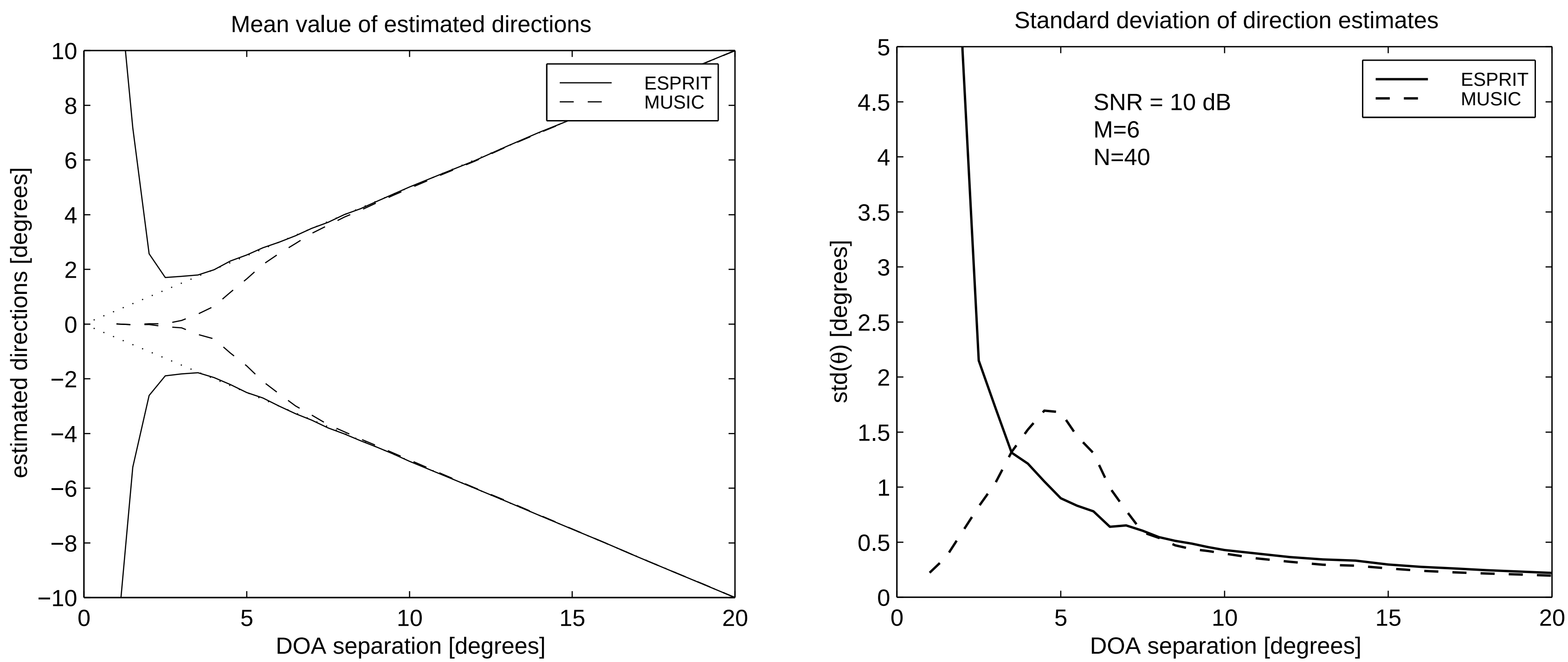




$d = 2$ sources, $M = 6$ antennas, $N = 40$ samples, $20^\circ$ separation.

Mean and standard deviations of ESPRIT and MUSIC as function of SNR

# The ESPRIT algorithm

## Performance (varying separation)




Mean and standard deviations of ESPRIT and MUSIC as function of DOA separation.

# Delay estimation using ESPRIT

## Principle

- Consider an FIR pulse shape function $g(t)$ that is oversampled by a factor $P$:

$$g(t) \quad \leftrightarrow \quad \mathbf{g}(0) = \begin{bmatrix} g(0) \\ g(\frac{1}{P}) \\ \vdots \\ g(L - \frac{1}{P}) \end{bmatrix}$$

$$g(t-\tau) \quad \leftrightarrow \quad \mathbf{g}(\tau) = \begin{bmatrix} g(0-\tau) \\ g(\frac{1}{P}-\tau) \\ \vdots \\ g(L - \frac{1}{P}-\tau) \end{bmatrix}$$

- Q: Given $\mathbf{g}(\tau)$ and knowing $\mathbf{g}(0)$, how do we estimate $\tau$?

  A: By using the fact that a Fourier transform maps a delay to a phase shift.

# Delay estimation using ESPRIT

- Apply the DFT to $\mathbf{g}(0)$:

$$\tilde{\mathbf{g}}(0) \;:=\; \mathcal{F}\mathbf{g}(0)\,, \qquad \mathcal{F} := \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W & \cdots & W^{LP-1} \\ \vdots & \vdots & & \vdots \\ 1 & W^{LP-1} & \cdots & W^{(LP-1)^2} \end{bmatrix}, \quad W = e^{-j\frac{2\pi}{LP}}$$

- The DFT of $\mathbf{g}(\tau)$ can then be written as

$$\tilde{\mathbf{g}}(\tau) := \mathcal{F}\mathbf{g}(\tau) \;=\; \tilde{\mathbf{g}}(0) \odot \begin{bmatrix} 1 \\ W^{\tau P} \\ (W^{\tau P})^2 \\ \vdots \\ (W^{\tau P})^{LP-1} \end{bmatrix} \;=\; \text{diag}(\tilde{\mathbf{g}}(0)) \cdot \begin{bmatrix} 1 \\ W^{\tau P} \\ (W^{\tau P})^2 \\ \vdots \\ (W^{\tau P})^{LP-1} \end{bmatrix}$$

( $\odot$ represents entrywise multiplication of the two vectors)

# Delay estimation using ESPRIT

- From the DFT of $\mathbf{g}(0)$ and $\mathbf{g}(\tau)$ we can compute

$$\mathbf{z} := \{\text{diag}(\tilde{\mathbf{g}}(0))\}^{-1}\tilde{\mathbf{g}}(\tau)$$

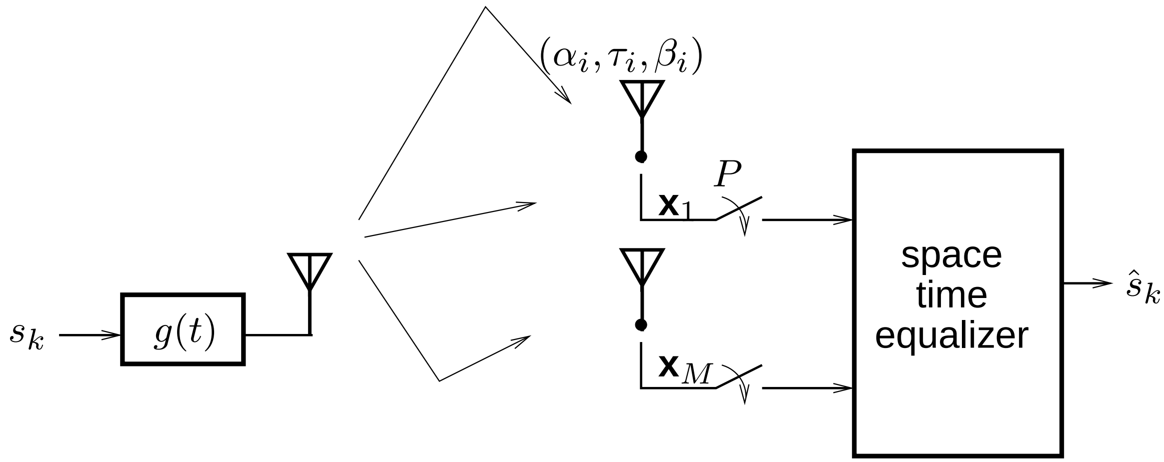
- The vector $\mathbf{z}$ has model

$$\mathbf{z} = \mathbf{f}(\phi), \qquad \mathbf{f}(\phi) := \begin{bmatrix} 1 \\ \phi \\ \phi^2 \\ \vdots \\ \phi^{LP-1} \end{bmatrix}, \qquad \phi := e^{j2\pi\tau/L}$$

- Now apply ESPRIT to compute $\phi$ and then $\tau$.

# Delay estimation using ESPRIT

## Multiple paths




- Consider a multipath channel which consists of $r$ delayed copies of $g(t)$

$$h(t) = \sum_{i=1}^{r} \beta_i g(t - \tau_i) \qquad \Leftrightarrow \qquad \mathbf{h} = \sum_{i=1}^{r} \mathbf{g}(\tau_i)\beta_i = [\mathbf{g}(\tau_1), \cdots, \mathbf{g}(\tau_r)] \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_r \end{bmatrix} =: \mathbf{G}\beta$$

- Assume pulse shape $g(t)$ is known and **h** has been estimated using training.

# Delay estimation using ESPRIT

- As before:

$$\mathbf{z} := \{\text{diag}(\tilde{\mathbf{g}}(0))\}^{-1}\mathcal{F}\mathbf{h}$$

- The vector $\mathbf{z}$ has model

$$\mathbf{z} = \mathbf{F}\beta, \qquad \mathbf{F} = [\mathbf{f}(\phi_1), \cdots, \mathbf{f}(\phi_r)], \qquad \mathbf{f}(\phi_i) := \begin{bmatrix} 1 \\ \phi_i \\ \phi_i^2 \\ \vdots \\ \phi_i^{LP-1} \end{bmatrix}, \qquad \phi_i := e^{j2\pi\tau_i/L}$$

- Conisder a shift of $\mathbf{z}$ and $\mathbf{f}(\phi_i)$ of length $LP-m+1$:

$$\mathbf{z}^{(l)} := \begin{bmatrix} z_l \\ z_{l+1} \\ \vdots \\ z_{l+LP-m} \end{bmatrix} \qquad \mathbf{f}(\phi_i)^{(l)} = \begin{bmatrix} \phi_i^l \\ \phi_i^{l+1} \\ \vdots \\ \phi_i^{l+LP-m} \end{bmatrix} = \begin{bmatrix} 1 \\ \phi_i \\ \vdots \\ \phi_i^{LP-m} \end{bmatrix} \phi_i^l =: \mathbf{f}'(\phi_i)\phi_i^l$$

# Delay estimation using ESPRIT

- Let us now stack the different shifts of **z**:

$$\mathbf{Z} = [\mathbf{z}^{(0)}, \mathbf{z}^{(1)}, \cdots, \mathbf{z}^{(m-1)}] = \mathbf{F}'\mathbf{B}', \qquad \mathbf{F}' = [\mathbf{f}'(\phi_1), \cdots, \mathbf{f}'(\phi_r)]$$

$$\mathbf{B}' = [\beta \quad \mathbf{\Phi}\beta \quad \mathbf{\Phi}^2\beta \quad \cdots \quad \mathbf{\Phi}^{m-1}\beta], \qquad \mathbf{\Phi} = \begin{bmatrix} \phi_1 & & \\ & \ddots & \\ & & \phi_r \end{bmatrix}$$

- Model: $\mathbf{X} = \mathbf{F}''\mathbf{B}', \quad \mathbf{Y} = \mathbf{F}''\mathbf{\Phi}\mathbf{B}'$.

- Now we can apply ESPRIT to estimate $\mathbf{\Phi}$ and hence all $\{\tau_i\}$.

  1. Compute the SVD: $\mathbf{Z} = \hat{\mathbf{U}}_z \hat{\Sigma}_z \hat{\mathbf{V}}_z^{\mathrm{H}}$
  2. Split $\hat{\mathbf{U}}_z$ into $\hat{\mathbf{U}}_x$ and $\hat{\mathbf{U}}_y$ (shift over 1 row): $\hat{\mathbf{U}}_z = \left[\begin{array}{c} \hat{\mathbf{U}}_x \\ \hline *** \end{array}\right] = \left[\begin{array}{c} *** \\ \hline \hat{\mathbf{U}}_y \end{array}\right]$

     Model: $\hat{\mathbf{U}}_x = \mathbf{F}''\mathbf{T}, \quad \hat{\mathbf{U}}_y = \mathbf{F}''\mathbf{\Phi}\mathbf{T}$ for some $r \times r$ matrix **T**.
  3. Compute the eigenvalue decomposition: $\hat{\mathbf{U}}_x^{\dagger}\hat{\mathbf{U}}_y = \mathbf{T}^{-1}\mathbf{\Phi}\mathbf{T}$.

# Delay estimation using ESPRIT

## Multiple paths, multiple antennas

- The channel vector $\mathbf{h}(t)$ with $M$ entries can then be written as

$$\mathbf{h}(t) = \sum_{i=1}^{r} \mathbf{a}(\theta_i)\beta_i g(t-\tau_i)$$

- Oversampling at rate $P$ we obtain

$$\mathbf{h} = \begin{bmatrix} \mathbf{h}(0) \\ \mathbf{h}(\frac{1}{P}) \\ \vdots \\ \mathbf{h}(L-\frac{1}{P}) \end{bmatrix} = \sum_{i=1}^{r} [\mathbf{g}(\tau_i) \otimes \mathbf{a}(\theta_i)]\beta_i \;=:\; [\mathbf{G} \circ \mathbf{A}]\beta$$

where

Kronecker product: $\mathbf{a} \otimes \mathbf{b} := \begin{bmatrix} a_1\mathbf{b} \\ \vdots \\ a_N\mathbf{b} \end{bmatrix}$

Khatri-Rao product: $\mathbf{A} \circ \mathbf{B} := [\mathbf{a}_1 \otimes \mathbf{b}_1 \quad \cdots \quad \mathbf{a}_r \otimes \mathbf{b}_r]$

# Delay estimation using ESPRIT

- Preprocess using DFT as before and construct **Z** from block-$M$ shifts:

$$\mathbf{Z} = [\mathbf{z}^{(0)}, \mathbf{z}^{(1)}, \cdots, \mathbf{z}^{(m-1)}] \qquad M(LP-m+1)\times m$$

- **Z** has a model

$$\mathbf{Z} = [\mathbf{F}' \circ \mathbf{A}]\mathbf{B}', \qquad \mathbf{F}' = [\mathbf{f}'(\phi_1), \cdots, \mathbf{f}'(\phi_r)], \qquad \mathbf{f}'(\phi_i) = \begin{bmatrix} 1 \\ \phi_i \\ \vdots \\ \phi_i^{LP-m} \end{bmatrix}$$

- Note

$$\mathbf{f}'(\phi_i) \otimes \mathbf{a}(\theta_i) = \begin{bmatrix} \mathbf{a}(\theta_i) \\ \phi_i \mathbf{a}(\theta_i) \\ \vdots \\ \phi_i^{LP-m} \mathbf{a}(\theta_i) \end{bmatrix}$$

- We can now apply the ESPRIT algorithm to **Z**, where **X** and **Y** are submatrices of **Z** with last/first $M$ rows omitted.

# Frequency estimation

## Use of ESPRIT to estimate frequencies

- Given that signal $x(t)$ is the sum of $d$ harmonic components, $x(t) = \sum_{i=1}^{d} \beta_i e^{j\omega_i t}$, estimate $\omega_i$ and $\beta_i$.

- Collect $N$ samples in a data matrix **Z** with $m$ rows:

$$\mathbf{Z} = \begin{bmatrix} x(0) & x(1) & \cdots & x(N-m-1) \\ x(1) & x(2) & \cdots & x(N-m) \\ \iddots & \iddots & & \iddots \\ x(m-1) & x(m) & \cdots & x(N-1) \end{bmatrix}$$

- **Z** has model

$$\mathbf{Z} = \mathbf{AS} := \begin{bmatrix} 1 & \cdots & 1 \\ \phi_1 & \cdots & \phi_d \\ \phi_1^2 & \cdots & \phi_d^2 \\ \vdots & & \vdots \end{bmatrix} \begin{bmatrix} \beta_1 & \beta_1\phi_1 & \beta_1\phi_1^2 & \cdots \\ \vdots & \vdots & & \\ \beta_d & \beta_d\phi_d & \beta_d\phi_d^2 & \cdots \end{bmatrix}, \qquad \phi_i = e^{j\omega_i}$$