An Algebraic Solution of the GPS Equations

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The global positioning system (GPS) equations are usually solved with an application of Newton's method or a variant thereof:

$$x_{n+1} = x_n + H^{-1}(t - f(x_n)). \tag{1}$$

Here x is a vector comprising the user position coordinates together with clock offset, t is a vector of four pseudorange measurements, and H is a measurement matrix of partial derivatives $H = f_x$. In fact the first fix of a Kalman filter provides a solution of this type. If more than four pseudoranges are available for extended batch processing, H^{-1} may be replaced by a generalized inverse $(H^{T}WH)^{-1}H^{T}W$, where W is a positive definite weighting matrix (usually taken to be the inverse of the measurement covariance matrix). This paper introduces a new method of solution that is algebraic and noniterative in nature, computationally efficient and numerically stable, admits extended batch processing, improves accuracy in bad geometric dilution of precision (GDOP) situations, and allows a "cold start" in deep space applications.

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ALGORITHMIC STATEMENT OF RESULTS

Let $\boldsymbol{x}$ and $\{s_i : 1 \le i \le n\}$ denote user and satellite position coordinates in any convenient Earth-centered cartesian coordinate system. Let $\{t_i : 1 \le i \le n\}$ denote the pseudorange measurements taken by the user from each of the n satellites:

$$t_i = d(\boldsymbol{x},\boldsymbol{s}_i) + b \tag{2}$$

where $d(\boldsymbol{x},\boldsymbol{y})$ is the distance from $\boldsymbol{x}$ to $\boldsymbol{y}$ and b is clock offset. Define the 1×4 column data vectors

$$\boldsymbol{a}_i = (\boldsymbol{s}_i^{T}\ t_i)^{T}, \qquad 1 \le i \le n \tag{3}$$

where T denotes the transpose. Define the Minkowski functional for 4-space by

$$\langle \boldsymbol{a},\boldsymbol{b} \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 - a_4 b_4. \tag{4}$$

Define

$$A = (\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \boldsymbol{a}_3 \quad \ldots \quad \boldsymbol{a}_n)^{T} \tag{5}$$

$$i_0 = (1 \quad 1 \quad 1 \quad \ldots \quad 1)^{T} \tag{6}$$

$$\boldsymbol{r} = (r_1 \quad r_2 \quad r_3 \quad \ldots \quad r_n)^{T} \tag{7}$$

where r_i, $1 \le i \le n$, is computed from

$$r_i = \langle \boldsymbol{a}_i,\boldsymbol{a}_i \rangle / 2. \tag{8}$$

Compute the generalized inverse

$$B = (A^{T}WA)^{-1}A^{T}W \tag{9}$$

where W is a symmetric positive definite weighting matrix. (The identity matrix will do. The matrix AB is an orthogonal projection operator in this case. For other choices of W oblique projections are obtained.) Compute the 1×4 column vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ from

$$\boldsymbol{u} = B\boldsymbol{i}_0 \tag{10}$$

$$\boldsymbol{v} = B\boldsymbol{r} \tag{11}$$

together with the scalar coefficients E,F,G defined by

$$E = \langle \boldsymbol{u},\boldsymbol{u} \rangle \tag{12}$$

$$F = \langle \boldsymbol{u},\boldsymbol{v} \rangle - 1 \tag{13}$$

$$G = \langle \boldsymbol{v},\boldsymbol{v} \rangle. \tag{14}$$

Solve the quadratic

$$E\,\lambda^2 + 2F\,\lambda + G = 0 \tag{15}$$

for the pair of roots $\lambda_{1,2}$. Compute the 1×4 column vectors $\boldsymbol{y}_{1,2}$ defined by

$$\boldsymbol{y}_{1,2} = \lambda_{1,2}\,\boldsymbol{u} + \boldsymbol{v}. \tag{16}$$

Then with the identification

$$\boldsymbol{y}^{T} = (\boldsymbol{x}^{T} \ - b)^{T} \tag{17}$$

either the pair $\boldsymbol{x}_1,b_1$ or the pair $\boldsymbol{x}_2,b_2$ will solve the GPS problem for user position and clock offset. To distinguish the actual solution, substitute back into the equations defining the original pseudoranges. There will be agreement in only one case.

SAMPLE COMPUTATION

Computations by hand are not very attractive without first making some gross simplifications. We assume a "one-dimensional" world with just two satellites present. Nonetheless the computations are illuminating. In particular a graphical interpretation of the "extraneous solution" is provided. Let satellite s_1 be placed at position -4 (Earth radius units) and satellite s_2 at position 4. Suppose pseudoranges 4 and 2 are observed. Then the user position x and clock offset b satisfy the pseudorange equations

$$|x + 4| + b = 4$$

$$|x - 4| + b = 2. \tag{18}$$

Recalling that $|x| =$ square root (x^2), we see that the equations are nonlinear, despite present appearances. The absolute value signs must be removed by a "squaring" process. This squaring process is responsible for the introduction of the extraneous solution. The situation has the simple graphical representation shown in Fig. 1.

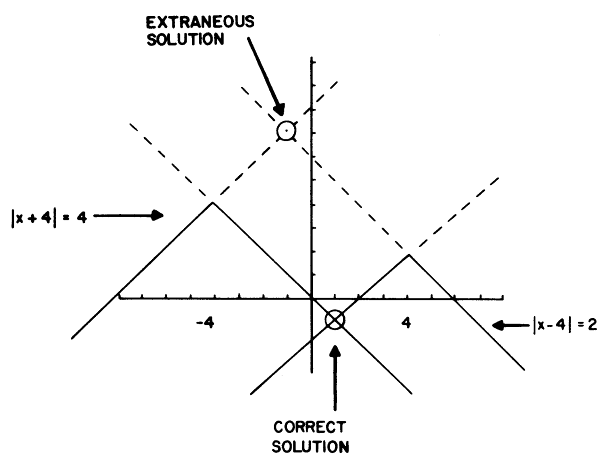

Fig. 1.

We follow our algorithm by constructing $A = (\boldsymbol{a}_1 \ \boldsymbol{a}_2)^{\mathrm{T}}$.

$$A = \begin{pmatrix} -4 & 4 \\ 4 & 2 \end{pmatrix}^{\mathrm{T}}. \tag{19}$$

The vector $\boldsymbol{i}_0$ is given by

$$\boldsymbol{i}_0 = (1 \quad 1)^{\mathrm{T}} \tag{20}$$

and $\boldsymbol{r} = (r_1 \ r_2)^{\mathrm{T}}$ with $r_i = \frac{1}{2} \langle a_i, a_i \rangle$, $1 \le i \le 2$, is given by

$$\boldsymbol{r} = (0 \quad 6)^{\mathrm{T}}. \tag{21}$$

Since A is square (for two satellites, two unknowns), the generalized inverse $B = (A^{\mathrm{T}}WA)^{-1}A^{\mathrm{T}}W$ reduces to A^{-1}, no matter what the positive definite matrix W:

$$B = \frac{1}{24}\begin{pmatrix} -2 & 4 \\ 4 & 4 \end{pmatrix}^{\mathrm{T}}. \tag{22}$$

Therefore $\boldsymbol{u} = B\boldsymbol{i}_0$ and $\boldsymbol{v} = B\boldsymbol{r}$ are given by

$$\boldsymbol{u} = (1/12 \quad 1/3)^{\mathrm{T}} \tag{23}$$

$$\boldsymbol{v} = (\ 1 \quad 1)^{\mathrm{T}}. \tag{24}$$

The scalars $E = \langle \boldsymbol{u},\boldsymbol{u} \rangle$, $F = \langle \boldsymbol{u},\boldsymbol{v} \rangle - 1$, and $G = \langle \boldsymbol{v},\boldsymbol{v} \rangle$ are given by

$$E = \quad 5/48 \tag{25}$$

$$F = -5/4 \tag{26}$$

$$G = \quad 0. \tag{27}$$

The roots $\lambda_{1,2}$ of $E\lambda^2 + 2F\lambda + G = 0$ are given by

$$\lambda_{1,2} = 0, \ -24. \tag{28}$$

Therefore $\boldsymbol{y}_{1,2} = \lambda_{1,2}\,\boldsymbol{u} + \boldsymbol{v}$ are given by

$$y_1 = \quad 0\,(1/12 \quad 1/3)^{\mathrm{T}} + (1 \quad 1)^{\mathrm{T}} \tag{29}$$

$$y_2 = -24\,(1/12 \quad 1/3)^{\mathrm{T}} + (1 \quad 1)^{\mathrm{T}}. \tag{30}$$

Through the identification $\boldsymbol{y} = (\boldsymbol{x}^{\mathrm{T}} \ -b)^{\mathrm{T}}$ this provides the two solution candidates

$$(x_1 \quad b_1)^{\mathrm{T}} = (\ 1 \quad -1)^{\mathrm{T}} \tag{31}$$

$$(x_2 \quad b_2)^{\mathrm{T}} = (-1 \quad 7)^{\mathrm{T}}. \tag{32}$$

By returning to the original pseudorange equations, the second of these is readily eliminated.

DISCUSSION OF RESULTS

Suppose $\boldsymbol{x}_0, b_0$ are solutions to the pseudorange equations

$$t_i = d(\boldsymbol{x}, \boldsymbol{s}_i) + b, \qquad 1 \le i \le n. \tag{33}$$

If we perturb these equations, we find that the perturbation variables δt, $\delta \boldsymbol{x}$, δb satisfy

$$\delta \boldsymbol{t} = (N^{\mathrm{T}} - \boldsymbol{i}_0)\begin{pmatrix} \delta \boldsymbol{x} \\ -\delta \boldsymbol{b} \end{pmatrix} \tag{34}$$

where $N = (\boldsymbol{n}_1^{\mathrm{T}}\ \boldsymbol{n}_2^{\mathrm{T}} \ldots \boldsymbol{n}_n^{\mathrm{T}})$, with $\boldsymbol{n}_i$ a unit vector pointing from the ith satellite $\boldsymbol{s}_i$ toward the position $\boldsymbol{x}_0$ (the gradient vector for $d(\boldsymbol{x}, \boldsymbol{s}_i)$). We can write this more compactly as

$$\delta \boldsymbol{t} = H\,\delta \boldsymbol{y}. \tag{35}$$

If the rank of H is four then we can multiply both sides through by $H^{\mathrm{T}}W$ and invert $H^{\mathrm{T}}WH$ to obtain

$$\delta \boldsymbol{y} = (H^{\mathrm{T}}WH)^{-1}H^{\mathrm{T}}W\,\delta \boldsymbol{t}. \tag{36}$$

The operator $H(H^{\mathrm{T}}WH)^{-1}H^{\mathrm{T}}W$ is a projection operator in the n-dimensional space of measurement error residuals. To see this we note that by inspection the operator is linear and idempotent, i.e., satisfies $PP = P$. The adjustment to $\boldsymbol{y}$ needed to minimize a weighted sum of error residuals δt_i, $1 \le i \le n$, is indicated by $\delta \boldsymbol{y}$. If W is the identity all errors are treated with equal importance. To gain further geometric insight, we note that the range of H is a subspace of the measurement space spanned by the columns of H. That particular combination of these columns that constructs the point in that range obtained by the above-described projection is afforded by $\delta \boldsymbol{y}$. The projection is orthogonal if $W = I_n$, the n-dimensional identity and in this case $\boldsymbol{y} + \delta \boldsymbol{y}$ is the usual "least squares" solution to the pseudorange equations. Whether or not we desire an orthogonal projection or an oblique projection (see Fig. 2) depends on our assessment of the relative importance of the various satellite pseudorange errors. Such matters as satellite health status, age of ephemeris, and signal-to-noise ratio (SNR) will determine this subjective weighting. If SNR is used as the sole criterion, we take W to be the inverse of the measurement covariance matrix.

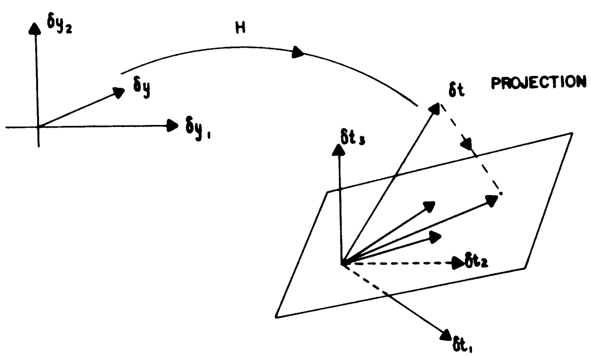

Fig. 2.

The covariance of our position/clock fix is described by the matrix

$$\text{cov} = (H^{\mathrm{T}} W H)^{-1}. \tag{37}$$

If W is taken to be the identity, then the trace of this matrix provides the GDOP number often used to determine the quality of the fix.

If when we collect our measured pseudoranges, we modify them with the clock offset adjustment

$$t_i := t_i - \text{avg} + R_{\text{orb}} \tag{38}$$

where

$$\text{avg} = \frac{1}{n} \sum_{i=1}^{n} t_i \tag{39}$$

and R_{orb} is the radius of the GPS satellite configuration (about 4 Earth radii, $(T_{\text{orb}}/T_{\text{Sch}})$ 2/3 to be exact, where T_{orb} is the period of a satellite and T_{Sch} is the Schuler frequency period) then the matrix A in our algorithm will reduce to the measurement matrix H if the user is positioned at the center of the Earth. The approximation is still quite good anywhere within the vicinity of the surface of the Earth. This can be verified directly by computer simulation, seen "geometrically" or analyzed by linear variational techniques. Thus in the vicinity of the surface of the Earth the matrix $(A^{\mathrm{T}} W A)^{-1}$ used in our algorithm affords a good approximation to the covariance of the solution provided by the algorithm as discussed above. This observation tells us to go ahead and pick W by the "usual" means.

Computer simulation shows that the algebraic solution performs better than an iterative solution in regions of poor GDOP. One can understand this result when it is realized that most iterative techniques improve a solution by replacing a nonlinear computation with a linear approximation, more or less using the first order terms in a Taylor series expansion of the nonlinearities. The algebraic solution can be regarded as a technique that uses the higher order terms in the series as well. Since the first-order terms tend to have degenerate rank in regions of poor GDOP, the higher order terms become critical in determining the nature of the singularity. Computer simulation shows that even with single precision arithmetic, user position can be determined with ¼ nm at the surface of the moon. Double precision arithmetic provides excellent accuracy. The region obviously has poor GDOP since the line of sight vectors pointing back toward the GPS constellation all tend to be collinear.

DEMONSTRATION OF THE RESULT

Through the artifice of introducing a few complex numbers the derivation of our algorithm can be made deceptively simple. (The original derivation was much more involved!)

The distance to the ith satellite is given by

$$d_i^2 = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} - 2\,\boldsymbol{s}_i^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{s}_i^{\mathrm{T}}\boldsymbol{s}_i, \qquad 1 \le i \le n. \tag{40}$$

The pseudoranges are

$$t_i = d_i + b, \qquad 1 \le i \le n. \tag{41}$$

Substituting (41) in (40),

$$t_i^2 - 2\,b\,t_i + b^2 = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} - 2\,\boldsymbol{s}_i^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{s}_i^{\mathrm{T}}\boldsymbol{s}_i. \tag{42}$$

Rearranging

$$\boldsymbol{s}_i^{\mathrm{T}}\boldsymbol{x} - t_i b = \tfrac{1}{2}(\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} - b^2) + \tfrac{1}{2}(\boldsymbol{s}_i^{\mathrm{T}}\boldsymbol{s}_i). \tag{43}$$

Define

$$L = \begin{pmatrix} 1 & & & \\ & 1 & & 0 \\ 0 & & 1 & \\ & & & j \end{pmatrix}; \quad L^{-1} = \begin{pmatrix} 1 & & & \\ & 1 & & 0 \\ 0 & & 1 & \\ & & & -j \end{pmatrix}. \tag{44}$$

Notice that $L = L^{\mathrm{T}}$ and $L^{-1} = L^{-\mathrm{T}}$. Also

$$L^2 = L^{-2} = \begin{pmatrix} 1 & & & \\ & 1 & & 0 \\ 0 & & 1 & \\ & & & -1 \end{pmatrix}. \tag{45}$$

Consulting (4) we see that

$$\langle \boldsymbol{a},\boldsymbol{b} \rangle = (L\boldsymbol{a})^{\mathrm{T}}(L\boldsymbol{b}) = (L^{-1}\boldsymbol{a})^{\mathrm{T}}(L^{-1}\boldsymbol{b}). \tag{46}$$

Define

$$\boldsymbol{z} = (\boldsymbol{x}^{\mathrm{T}}\ b)^{\mathrm{T}} \tag{47}$$

and

$$\lambda = \tfrac{1}{2}\langle \boldsymbol{z},\boldsymbol{z} \rangle. \tag{48}$$

Then (43) can be written as

$$\langle a_i, z\rangle = \lambda + r_i, \qquad 1 \le i \le n \tag{49}$$

where r_i is given in (8). From (46) we can write this as

$$(La_1)^T(Lz) = \lambda + r_1$$
$$(La_2)^T(Lz) = \lambda + r_2$$
$$\vdots$$
$$(La_n)^T(Lz) = \lambda + r_n \tag{50}$$

or, consulting the definitions (5), (6), and (7)

$$(LA^T)^T(Lz) = \lambda i_0 + r. \tag{51}$$

Equation (51) can be written in the form $\langle A^T, z\rangle = \lambda i_0 + r$, provided that $\langle A^T, z\rangle$ is defined by the left-hand side expression of that equation. In any event, we can start our matrix manipulations directly from (51): $AL^2z = \lambda i_0 + r$, $A^TWAL^2z = A^TW(\lambda i_0 + r)$, $L^2z = (A^TWA)^{-1}A^TW(\lambda i_0 + r)$, $Lz = L^{-1}B(\lambda i_0 + r)$, and finally

$$Lz = L^{-1}(\lambda u + v) \tag{52}$$

where we have made use of definitions (9), (10), and (11). Substituting (52) into (46) we see that λ satisfies

$$\langle z, z\rangle = \langle u, u\rangle \lambda^2 + 2\langle u, v\rangle \lambda + \langle v, v\rangle \tag{53}$$

which with definitions (12), (13), (14), and (48) give

$$E\lambda^2 + 2F\lambda + G = 0. \tag{54}$$

From (52) we also have

$$z = L^{-2}(\lambda u + v). \tag{55}$$

From (17), (45), and (47) we may write

$$y = L^2 z. \tag{56}$$

This together with (55) produces

$$y = \lambda u + v. \tag{57}$$

This completes the demonstration.

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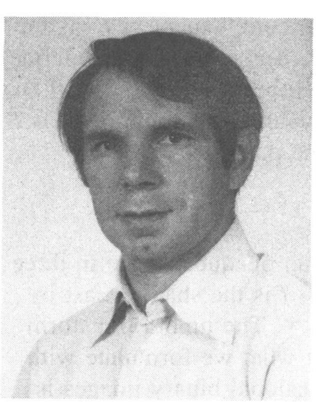

Stephen Bancroft received his B.A. in Physics from Amherst College in 1965 and the Ph.D. in Applied Mathematics from Brown University in 1968. Continuing studies include Qualitative Theory of Ordinary Differential Equations, Optimal Estimation, Inertial Guidance Systems, Navigation, Operations Research, and New Testament Greek. For the past 7 years Dr. Bancroft has been working at King Radio in Olathe, Ks. designing general aviation navigation systems. He has also been teaching part time at Mid-America Nazarene College in Olathe, Ks.