ET4350: Applied Convex Optimization

Delft University of Technology

Course Information

- Book(s) are freely available online
  - Stephen Boyd and Lieven Vandenberghe, "Convex Optimization", Cambridge University Press, 2004.
  - Slides/lecture notes for subgradient methods.
- Assessment
  - Open-book written exam.
  - Compulsory lab assignment worth 1 EC (20%); report and short presentation. Enroll via Brightspace.
- Course information:
  - http://ens.ewi.tudelft.nl/Education/courses/ee4530/index.php

Mathematical optimization

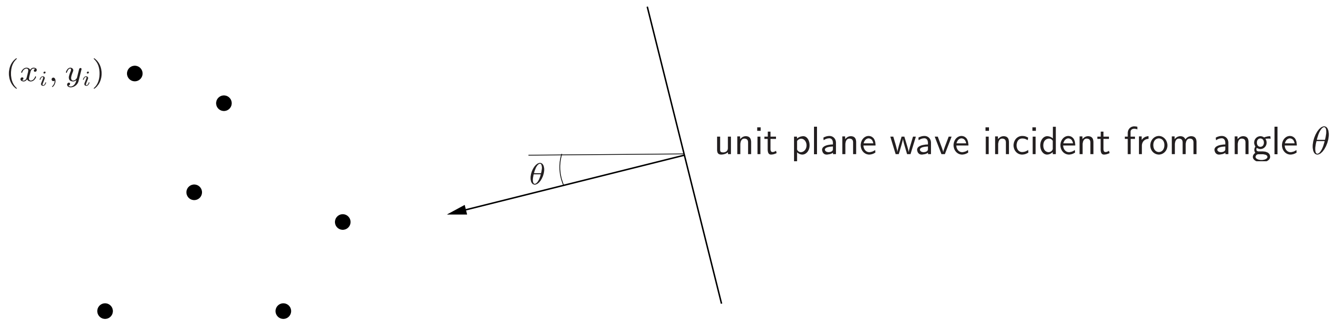
(mathematical) optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq b_i, \quad i = 1, \ldots, m \end{array}$$

- $x = (x_1, \ldots, x_n)$: optimization variables
- $f_0 : \mathbf{R}^n \to \mathbf{R}$: objective function
- $f_i : \mathbf{R}^n \to \mathbf{R}$, $i = 1, \ldots, m$: constraint functions

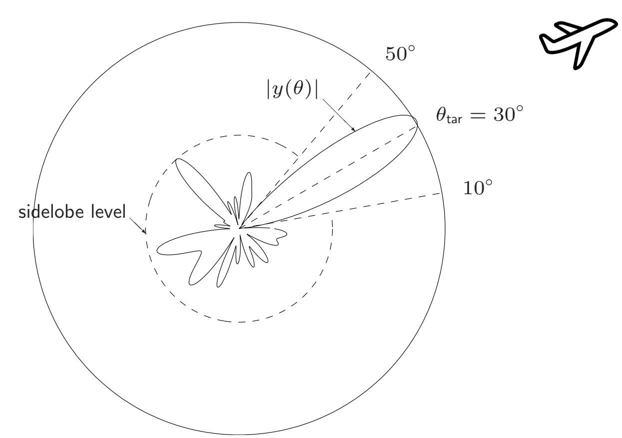
optimal solution $x^\star$ has smallest value of f_0 among all vectors that satisfy the constraints

Array processing



- omnidirectional antenna elements at positions $(x_1, y_1), \ldots, (x_n, y_n)$
- linearly combine with complex weights w_i:

$$y(\theta) = \sum_{i=1}^{n} w_i e^{j(x_i \cos\theta + y_i \sin\theta)}$$



- $y(\theta)$ is (complex) *antenna array gain pattern*
- $|y(\theta)|$ gives sensitivity of array as function of incident angle θ
- depends on design variables **Re** w, **Im** w
  (called *antenna array weights* or *shading coefficients*)

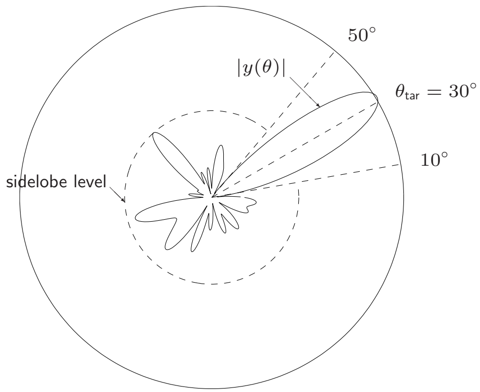
design problem: choose w to achieve desired gain pattern

Array processing

Sidelobe level minimization

make $|y(\theta)|$ small for $|\theta - \theta_{\text{tar}}| > \alpha$

(θ_{tar}: target direction; 2α: beamwidth)



via least-squares (discretize angles)

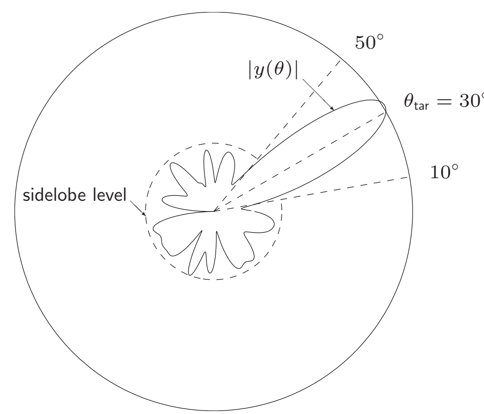
$$\begin{array}{ll} \text{minimize} & \sum_i |y(\theta_i)|^2 \\ \text{subject to} & y(\theta_{\text{tar}}) = 1 \end{array}$$

(sum is over angles outside beam)

minimize sidelobe level (discretize angles)

$$\begin{array}{ll} \text{minimize} & \max_i |y(\theta_i)| \\ \text{subject to} & y(\theta_{\text{tar}}) = 1 \end{array}$$

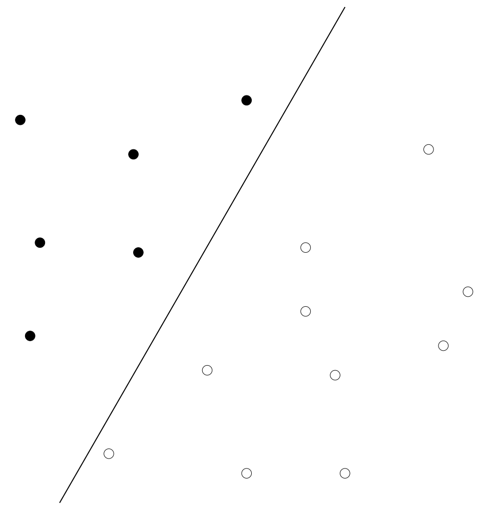
(max over angles outside beam)



Machine learning

separate two sets of points $\{x_1, \ldots, x_N\}$, $\{y_1, \ldots, y_M\}$ by a hyperplane:

$$a^T x_i + b > 0, \quad i = 1, \ldots, N, \qquad a^T y_i + b < 0, \quad i = 1, \ldots, M$$

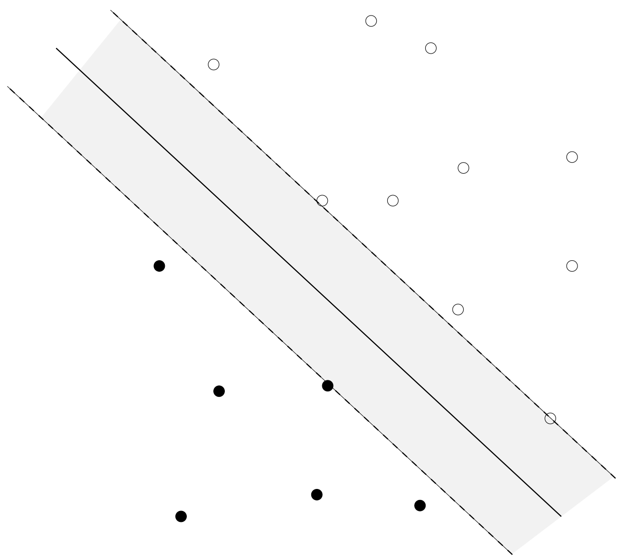


homogeneous in a, b, hence equivalent to

$$a^T x_i + b \geq 1, \quad i = 1, \ldots, N, \qquad a^T y_i + b \leq -1, \quad i = 1, \ldots, M$$

a set of linear inequalities in a, b

Machine learning



(Euclidean) distance between hyperplanes

$$\begin{aligned}\mathcal{H}_1 &= \{z \mid a^T z + b = 1\} \\ \mathcal{H}_2 &= \{z \mid a^T z + b = -1\}\end{aligned}$$

is $\mathbf{dist}(\mathcal{H}_1, \mathcal{H}_2) = 2/\|a\|_2$

to separate two sets of points by maximum margin,

$$\begin{array}{ll}\text{minimize} & (1/2)\|a\|_2 \\ \text{subject to} & a^T x_i + b \geq 1, \quad i = 1, \ldots, N \\ & a^T y_i + b \leq -1, \quad i = 1, \ldots, M\end{array}$$

Examples

portfolio optimization

- variables: amounts invested in different assets
- constraints: budget, max./min. investment per asset, minimum return
- objective: overall risk or return variance

device sizing in electronic circuits

- variables: device widths and lengths
- constraints: manufacturing limits, timing requirements, maximum area
- objective: power consumption

data fitting

- variables: model parameters
- constraints: prior information, parameter limits
- objective: measure of misfit or prediction error

Solving optimization problems

general optimization problem

- very difficult to solve
- methods involve some compromise, $e.g.$, very long computation time, or not always finding the solution

exceptions: certain problem classes can be solved efficiently and reliably

- least-squares problems
- linear programming problems
- convex optimization problems

Least-squares

$$\text{minimize} \quad \|Ax - b\|_2^2$$

solving least-squares problems

- analytical solution: $x^\star = (A^TA)^{-1}A^Tb$
- reliable and efficient algorithms and software
- computation time proportional to n^2k ($A \in \mathbf{R}^{k\times n}$); less if structured
- a mature technology

using least-squares

- least-squares problems are easy to recognize
- a few standard techniques increase flexibility ($e.g.$, including weights, adding regularization terms)

Linear programming

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \end{array}$$

solving linear programs

- no analytical formula for solution
- reliable and efficient algorithms and software
- computation time proportional to $n^2 m$ if $m \geq n$; less with structure
- a mature technology

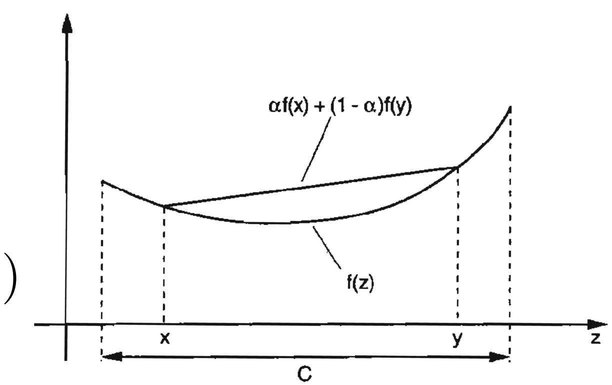
using linear programming

- not as easy to recognize as least-squares problems
- a few standard tricks used to convert problems into linear programs ($e.g.$, problems involving ℓ_1- or ℓ_∞-norms, piecewise-linear functions)

Convex optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le b_i, \quad i = 1, \ldots, m \end{array}$$

- objective and constraint functions are convex:

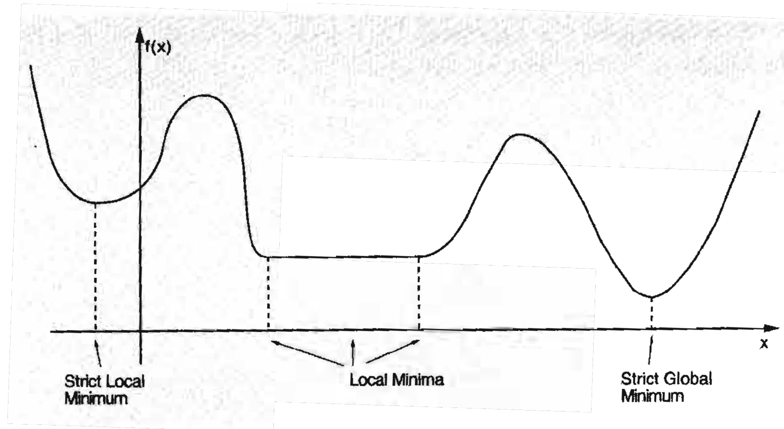


$$f_i(\alpha x + \beta y) \le \alpha f_i(x) + \beta f_i(y)$$

if $\alpha + \beta = 1$, $\alpha \ge 0$, $\beta \ge 0$

- includes least-squares problems and linear programs as special cases

The case of a convex cost function



Local minima: $x^\star$ is an unconstrained local minimum of $f_0 : \mathbf{R}^n \mapsto \mathbf{R}$ if is no worse than its neighbors.

$$f_0(x^\star) \leq f_0(x), \quad \forall x \in \mathbf{R}^n \text{ with } \|x - x^\star\| < \epsilon$$

for $\epsilon > 0$.

Global minima: $x^\star$ is an unconstrained local minimum of $f_0 : \mathbf{R}^n \mapsto \mathbf{R}$ if it is no worse than all other vectors.

$$f_0(x^\star) \leq f_0(x), \quad \forall x \in \mathbf{R}^n.$$

When the function is convex every local minimum is also global.

solving convex optimization problems

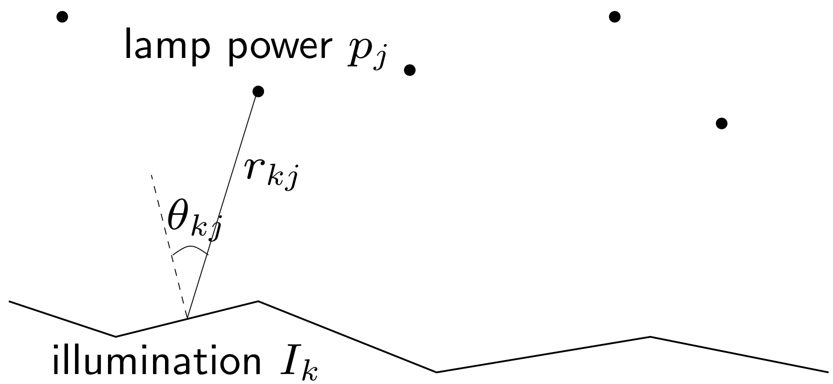
- no analytical solution
- reliable and efficient algorithms
- computation time (roughly) proportional to $\max\{n^3, n^2m, F\}$, where F is cost of evaluating f_i's and their first and second derivatives
- almost a technology

using convex optimization

- often difficult to recognize
- many tricks for transforming problems into convex form
- surprisingly many problems can be solved via convex optimization

Example

m lamps illuminating n (small, flat) patches



intensity I_k at patch k depends linearly on lamp powers p_j:

$$I_k = \sum_{j=1}^{m} a_{kj} p_j, \qquad a_{kj} = r_{kj}^{-2} \max\{\cos\theta_{kj}, 0\}$$

problem: achieve desired illumination I_{des} with bounded lamp powers

$$\begin{array}{ll} \text{minimize} & \max_{k=1,\ldots,n} |\log I_k - \log I_{\text{des}}| \\ \text{subject to} & 0 \leq p_j \leq p_{\text{max}}, \quad j = 1, \ldots, m \end{array}$$

how to solve?

1. use uniform power: $p_j = p$, vary p
2. use least-squares:

$$\text{minimize} \quad \sum_{k=1}^{n} (I_k - I_{\text{des}})^2$$

   round p_j if $p_j > p_{\max}$ or $p_j < 0$

3. use weighted least-squares:

$$\text{minimize} \quad \sum_{k=1}^{n} (I_k - I_{\text{des}})^2 + \sum_{j=1}^{m} w_j (p_j - p_{\max}/2)^2$$

   iteratively adjust weights w_j until $0 \leq p_j \leq p_{\max}$

4. use linear programming:

$$\begin{array}{ll} \text{minimize} & \max_{k=1,\ldots,n} |I_k - I_{\text{des}}| \\ \text{subject to} & 0 \leq p_j \leq p_{\max}, \quad j = 1, \ldots, m \end{array}$$

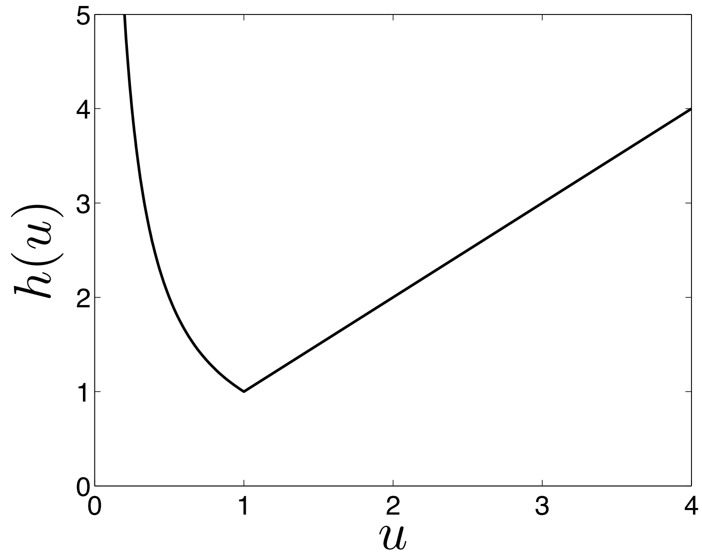
   which can be solved via linear programming

of course these are approximate (suboptimal) 'solutions'

5. use convex optimization: problem is equivalent to

$$\begin{array}{ll} \text{minimize} & f_0(p) = \max_{k=1,\ldots,n} h(I_k/I_{\text{des}}) \\ \text{subject to} & 0 \leq p_j \leq p_{\text{max}}, \quad j = 1, \ldots, m \end{array}$$

with $h(u) = \max\{u, 1/u\}$



f_0 is convex because maximum of convex functions is convex

exact solution obtained with effort ≈ modest factor × least-squares effort

additional constraints: does adding 1 or 2 below complicate the problem?

1. no more than half of total power is in any 10 lamps
2. no more than half of the lamps are on ($p_j > 0$)

- answer: with (1), still easy to solve; with (2), extremely difficult
- moral: (untrained) intuition doesn't always work; without the proper background very easy problems can appear quite similar to very difficult problems

Course goals and topics

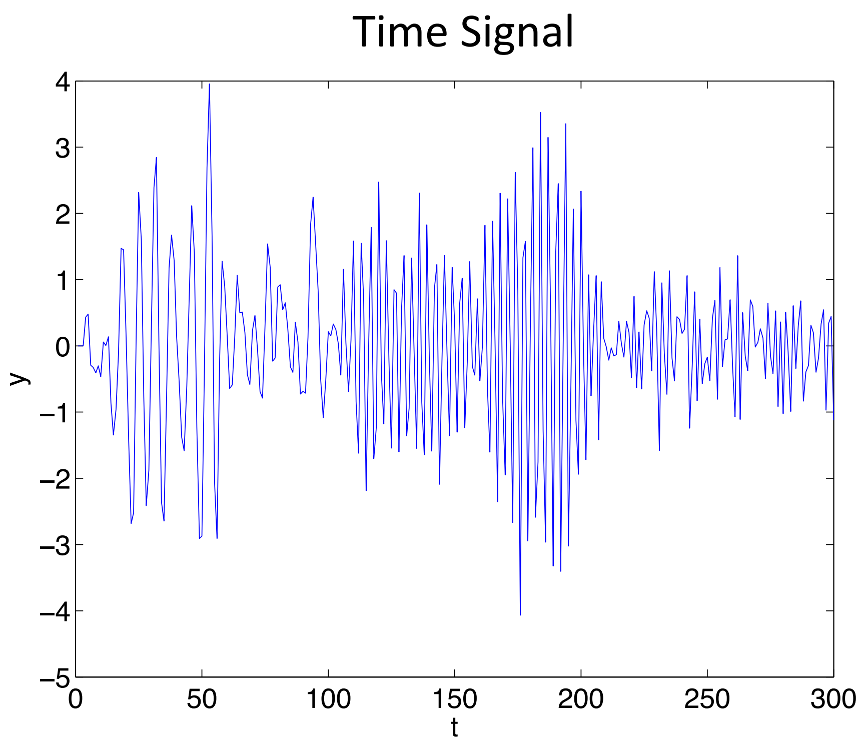
Goals

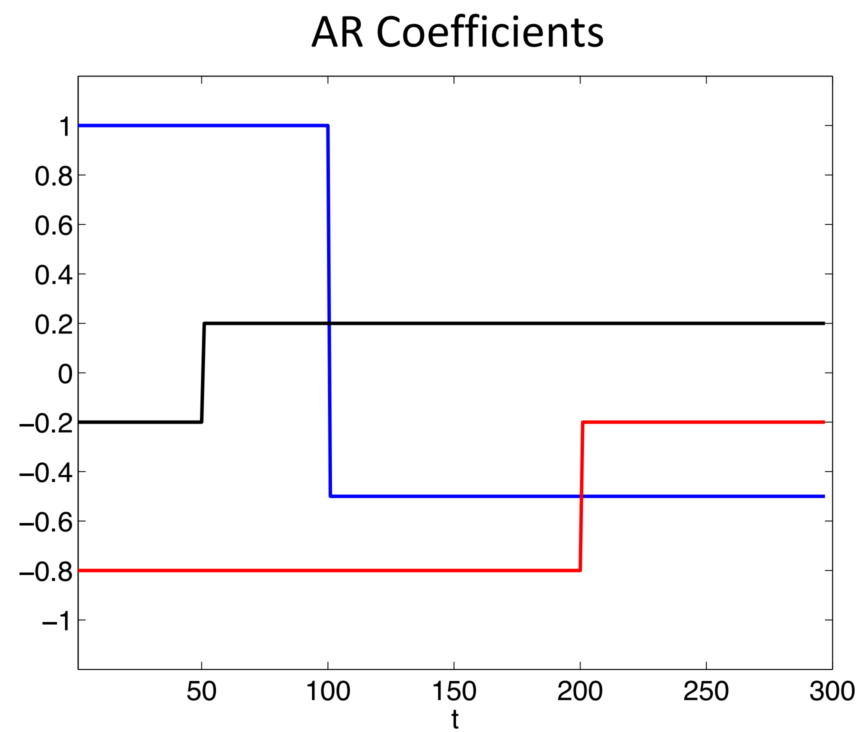
1. recognize and formulate problems (such as the illumination problem, classification, etc.) as convex optimization problems
2. Use optimization tools (CVX, YALMIP, etc.) as a part the lab assignment.
3. characterize optimal solution (optimal power distribution), give limits of performance, etc.

Topics

1. Background and optimization basics;
2. Convex sets and functions;
3. Canonical convex optimization problems (LP, QP, SDP);
4. Second-order methods (unconstrained and constrained optimization);
5. First-order methods (gradient, subgradient);

Project 1: Change Detection in Time Series Model

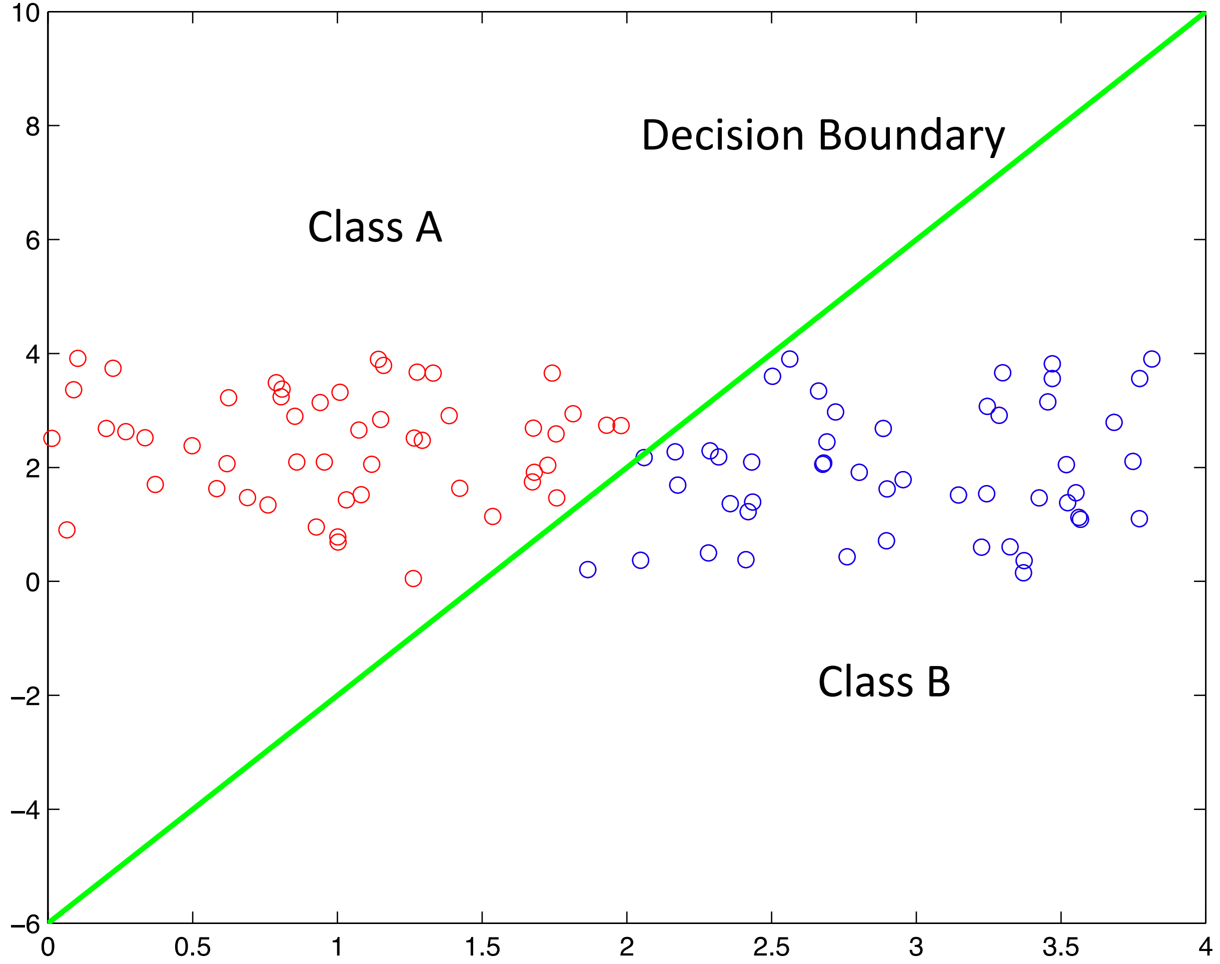




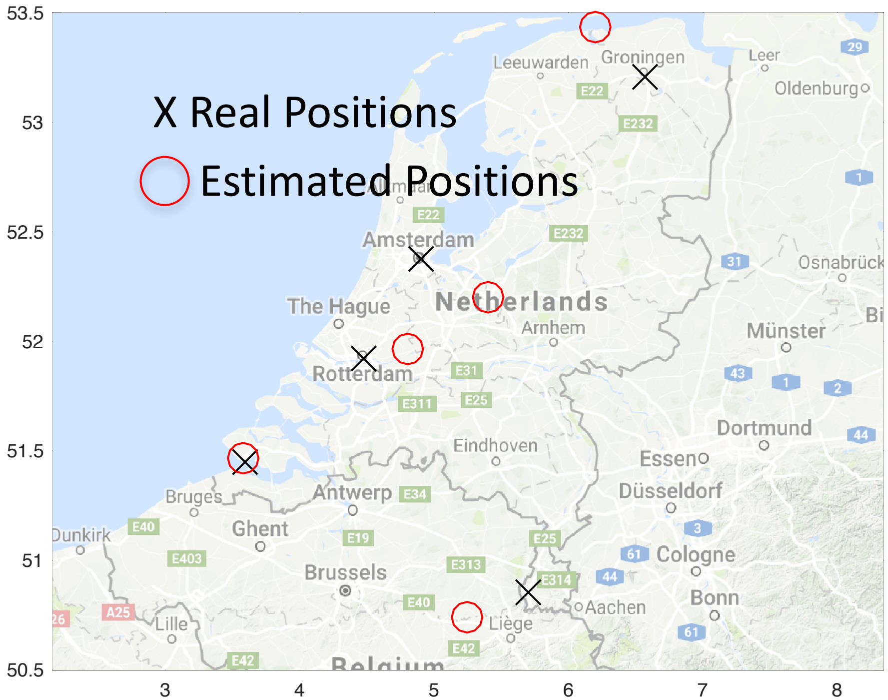
$$y(t+3) = a(t)y(t+2) + b(t)y(t+1) + c(t)y(t) + v(t); \quad v(t) \sim \mathcal{N}(0, 0.5^2).$$

assumption: $a(t)$, $b(t)$, and $c(t)$ are piecewise constant, change infrequently

Project 2: Linear Support Vector Machines



Project 3: Multidimensional Scaling for Localization.

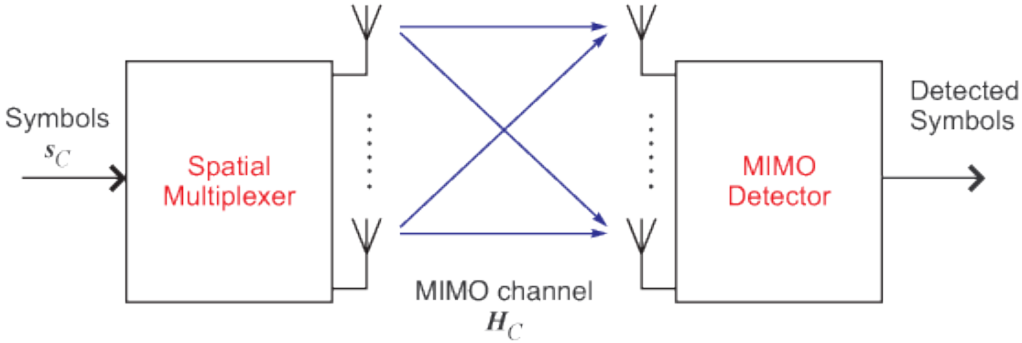


	A	R	M	V	G
A	0	71	146	177	127
R	71	0	136	104	159
M	146	136	0	208	258
V	177	104	208	0	279
G	127	159	258	279	0

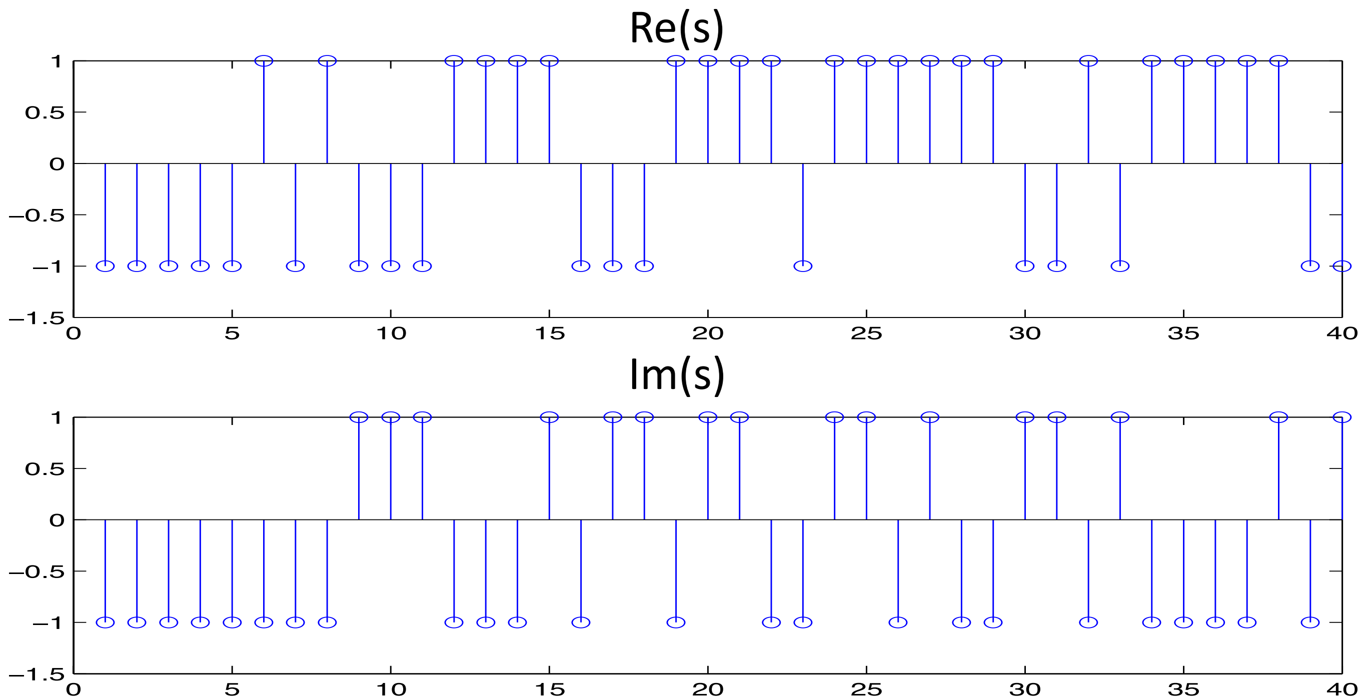
$$t_{ij}^2 \propto \|\mathbf{x}_i - \mathbf{x}_j\|^2$$

$$\mathbf{T} = \mathbf{1}\mathrm{diag}(\mathbf{X}^T\mathbf{X}) - 2\mathbf{X}^T\mathbf{X} + \mathrm{diag}(\mathbf{X}^T\mathbf{X})\mathbf{1}^T$$

Project 4: MIMO Detection



$$\mathbf{y}_c = \mathbf{H}_c \mathbf{s}_c + \mathbf{v}_c.$$



entries of $\mathbf{s}_c$ belong to the finite-alphabet set $\{\pm 1 \pm j\}$

Project 5: Compressed Sensing