# Applied Convex Optimization, EE4530, 2015

## Homework Set 5

**Exercise 1** [2pt.]
Solve Exercise 5.30 of Boyd, Vandenberghe, CO.

**Exercise 2.** [2pt.]
Consider the problem

$$\underset{\mathbf{x}}{\text{minimize}}\; f(\mathbf{x}) \quad \text{subject to} \quad g(\mathbf{x}) \leq 0,\, \mathbf{x} \in X,$$

where $\mathbf{x} \in \mathbf{R}^n$, $f : \mathbf{R}^n \to \mathbf{R}$, $g : \mathbf{R}^n \to \mathbf{R}$, and $X$ is a compact set. Call $f^*$ the infimum of the primal problem, and $q^*$ the supremum of the dual problem.

(a) When is the problem convex?

(b) Show that the dual function $q(\lambda)$ of the problem is

$$q(\lambda) = \min_{\mathbf{x} \in X} \{f(\mathbf{x}) + \lambda g(\mathbf{x})\}.$$

Show that its supremum $q^*$ is a lower bound for the infimum $f^*$ (weak duality). When does $q^* = f^*$?

(c) In the convex case, by leveraging Danskin's Theorem (you can find it on wikipedia, or in Proposition B.25 of Bertsekas, *Nonlinear Programming*) show that the sub-differential set of $q(\lambda)$ is

$$\mathrm{conv}\{g(\mathbf{x}^*(\lambda))\},$$

where conv represents the convex hull, while $\mathbf{x}^*(\lambda)$ is any optimizer of

$$\min_{\mathbf{x} \in X} \{f(\mathbf{x}) + \lambda g(\mathbf{x})\}.$$

(d) Apart from convexity, under which conditions on $f$, $g$, and $X$, the dual function $q(\lambda)$ is differentiable?

(e) Under the condition of (d) devise a gradient method to find $q^*$ (aka, to solve the dual problem). You may consider using a projected gradient method (as explained in the Notes, pages 14-15).

**Exercise 3.** [3pt.]
Consider the convex problem

$$\underset{\mathbf{x} \in [0,1]^{10}}{\text{minimize}} \sum_{i=1}^{10} \sigma_i \log(1 + x_i) \quad \text{subject to} \quad \sum_{i=1}^{10} x_i \leq 1,$$

where $\sigma_i$ is drawn from a uniform distribution in $[-1, 0]$.

(a) Argue that Slater's condition holds.

(b) Show that the Lagragian $L(\mathbf{x}, \lambda)$ is separable in $\mathbf{x}$ and so it is the dual function. I.e., show that

$$q(\lambda) = \sum_{i=1}^{10} q_i(\lambda),$$

for certain local dual functions $q_i(\lambda)$.

(c) *(Matlab)* For a given random instance of the primal problem, solve the dual problem with the gradient method you have devised in Exercise 2. Compare the $q^*$ with the $f^*$ you can obtain by solving the primal problem via Yalmip/SeDuMi (or CVX).

**Exercise 4.** [3pt.]
*(Matlab)* By using Yalmip/SeDuMi (or CVX) extract both the primal $\mathbf{x}^*$ and the dual $\lambda^*$ of the convex problem in Exercise 3. Compare them with the ones you can obtain with your gradient method. Plot the quantities $\|\mathbf{x}[k] - \mathbf{x}^*\|$ and $\|\lambda[k] - \lambda^*\|$ w.r.t. the iteration $k$ of your gradient scheme and comment on the convergence.