Applied Convex Optimization, EE4530, 2015

Homework Set 1

Exercise 1. [0pt., but if wrong or not done −2pt.]

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a sufficiently smooth function (that is, f can be derived an arbitrary number of times).

(a) For the case $n = 2$, consider a function $f(\mathbf{x})$, where $\mathbf{x}$ is a vector with two real components x_1 and x_2: compute the first, second, and third order derivatives of f with respect to $\mathbf{x}$;

(b) generalize the previous results for generic n;

(c) applied the previous results to $f(\mathbf{x}) = \sin(x_1) + \cos(x_1 x_2) - \tan(x_1)\exp(x_3)$.

Exercise 2. [3pt.]

(a) Find all local minima of the 2-dimensional function $f(x, y) = \frac{1}{2}x^2 + x\cos y$;

(b) Find the rectangular parallelepiped of unit volume that has the minimum surfact area.

Hint: By eliminating one of the dimensions, show that the problem is equivalent to the minimization over $x > 0$ and $y > 0$ of

$$f(x, y) = xy + \frac{1}{x} + \frac{1}{y}. \tag{1}$$

Exercise 3. [3pt.]

Suppose f is quadratic and of the form $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} - \mathbf{b}^T\mathbf{x}$ where $\mathbf{Q}$ is positive definite and symmetric.

(a) Show that the Lipschitz condition $\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\|$ is satisfied with L equal to the maximal eigenvalue of $\mathbf{Q}$.

Hint: Use the fact that for positive definite matrices $\|\mathbf{Q}(\mathbf{x}-\mathbf{y})\| \leq \lambda_{\max}(\mathbf{Q})\|\mathbf{x}-\mathbf{y}\|$, where $\lambda_{\max}(\mathbf{Q})$ is the maximal eigenvalue of $\mathbf{Q}$.

(b) Consider the gradient method $\mathbf{x}[k+1] = \mathbf{x}[k] - \alpha\mathbf{D}\nabla f(\mathbf{x}[k])$, where $\mathbf{D}$ is positive definite and symmetric. Show that the method converges to $\mathbf{x}^* = \mathbf{Q}^{-1}\mathbf{b}$ for every starting point $\mathbf{x}[0]$ if and only if $\alpha \in (0, 2/\bar{L})$, where $\bar{L}$ is the maximum eigenvalue of $\mathbf{D}^{1/2}\mathbf{Q}\mathbf{D}^{1/2}$.

Hint: Write $f(\mathbf{x}[k+1])$ in terms of $f(\mathbf{x}[k])$ by using a Taylor expansion, and show that if and only if $\alpha \in (0, 2/\bar{L})$ then the sequence $\{f(x[k])\}$ is monotonically decreasing. Find the limit point of such a sequence and prove its uniqueness.

Exercise 4. [4pt.]

(Matlab) It is given the function,

$$f(x_1, x_2) = (1 - x_1)^2 + 100(x_2 - x_1^2)^2$$

(a) Plot the function. Is it convex?;

(b) Find a minimum by programming in Matlab a gradient method. Is it the global minimum? (Can you find the global minimum analytically?);

(c) Find a minimum by programming in Matlab a Newton's method. Is it the global minimum?;

(d) Find different local solutions by changing the initial conditions. Is the gradient method faster or slower to converge w.r.t. the Newton method? (Plot convergence w.r.t. iterations).

(e) Compare your programs with the Matlab function *fminunc*.