EE3S1 Signal Processing – DSP

Lecture 4: Discrete Fourier Transform (Ch. 10)

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- Sampling in frequency
- Definition of the DFT, derivation of the Inverse DFT
- Aliasing
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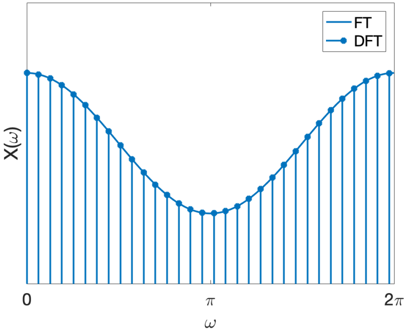


- Properties:
  - Zero padding (interpolation property) [Lecture 5]
  - Cyclic vs. linear convolution
  - Fast computation using the FFT [Lecture 7]

Sampling in frequency

Even if $x[n]$ is time-discrete, $X(\omega)$ is continuous in ω.

- But if we plot $X(\omega)$, we can plot only samples of it.

Suppose we plot only N samples, uniformly spaced on $0, \cdots, 2\pi$:

$$X[k] := X\left(\frac{2\pi}{N}k\right), \qquad k = 0, \cdots, N-1$$

- How are these N samples related to the samples of $x[n]$? Of course,

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\frac{2\pi}{N}kn}$$

  But can we do with just N samples in time as well?

- By duality, we expect sampling $\Leftrightarrow$ periodicity
  And as a consequence, issues around aliasing...

Sampling in frequency

What do these N frequency samples tell us about $x[n]$?

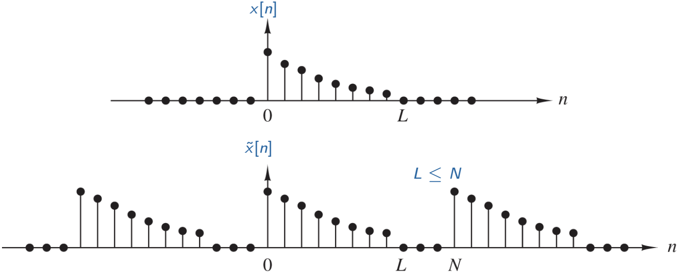
$$
\begin{aligned}
X[k] = X\left(\tfrac{2\pi}{N}k\right) &= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\frac{2\pi}{N}kn} \\
&= \sum_{m=-\infty}^{\infty} \sum_{n=mn}^{(m+1)N-1} x[n]\, e^{-j\frac{2\pi}{N}kn} \\
&= \sum_{m=-\infty}^{\infty} \sum_{n=0}^{N-1} x[n+mN]\, e^{-j\frac{2\pi}{N}kn} \\
&= \sum_{n=0}^{N-1} \underbrace{\left(\sum_{m=-\infty}^{\infty} x[n+mN] \right)}_{\tilde{x}[n]} e^{-j\frac{2\pi}{N}kn}
\end{aligned}
$$

Thus, these N frequency samples are defined by a DTFT of N samples (one period) of a periodic signal $\tilde{x}[n]$.

Sampling in frequency

Define $\tilde{x}[n] = \sum_{m=-\infty}^{\infty} x[n + mN]$.

- Gives rise to temporal aliasing
- Periodic, defined by N samples in the "fundamental interval" (in time) $0, \cdots, N-1$
- If $x[n]$ has length $L \leq N$, then from $\tilde{x}[n]$ we can recover $x[n]$ by windowing. Otherwise, destructive aliasing.



Towards the DFT

- As shown on slide 4, we have from the DTFT

$$\textbf{DFT:} \quad X[k] = \sum_{n=0}^{N-1} \tilde{x}[n]\, e^{-j\frac{2\pi}{N}kn}\,, \qquad k = 0, \cdots, N-1$$

  Thus, N samples in time are related to N samples in frequency.

- Can we invert this relation? Yes, we will show that

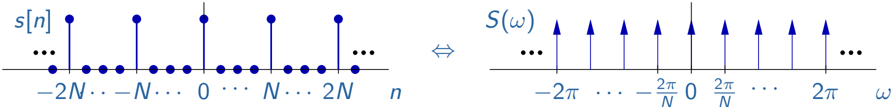
$$\textbf{IDFT:} \quad \tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}\,, \qquad n = 0, \cdots, N-1$$

(Alternative viewpoint, explored later: the DFT is related to an $N \times N$ matrix, and the IDFT to the inverse of that matrix.)

Proof (4 pages)

Define the sample function $s[n]$ (delta train*) in time. The next page shows that its corresponding DTFT $S(\omega)$ is a delta train in frequency:

$$s[n] = \sum_m \delta[n - mN] \quad \Leftrightarrow \quad S(\omega) = \frac{2\pi}{N} \sum_k \delta(\omega - \tfrac{2\pi}{N} k)$$



(*Book writes $\tilde{\delta}[n]$ instead of $s[n]$.)

Proof (2)

To show this, first derive

$$\frac{1}{N}\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}kn} = \frac{1}{N}\frac{1-e^{j\frac{2\pi}{N}nN}}{1-e^{j\frac{2\pi}{N}n}} = \begin{cases}1, & n = 0, \pm N, \pm 2N, \cdots \\ 0, & \text{otherwise}\end{cases} = s[n]$$

(viz. Fourier Series). Then the DTFT is (for $0 \leq \omega < 2\pi$)

$$S(\omega) = \mathcal{F}\{s[n]\} = \frac{1}{N}\sum_{k=0}^{N-1}\mathcal{F}\{e^{j\frac{2\pi}{N}kn}\} = \frac{2\pi}{N}\sum_{k=0}^{N-1}\delta(\omega - \tfrac{2\pi}{N}k)$$

Outside the fundamental interval, the spectrum is periodic; for this, extend the sum over all $k = -\infty$ to ∞.

We used:

$$1 + x + x^2 + \cdots + x^{N-1} = \frac{1-x^N}{1-x}$$

Proof (3)

- Sample in frequency (product with delta train):

$$\begin{aligned}\tilde{X}(\omega) &= X(\omega)\, S(\omega) \\ &= \frac{2\pi}{N} \sum_k X(\tfrac{2\pi}{N} k)\delta(\omega - \tfrac{2\pi}{N} k)\end{aligned}$$

  On the interval $0 \leq \omega < 2\pi$, we have just N samples $X[k] = X(\frac{2\pi}{N} k)$, $k = 0, \cdots, N-1$, scaled by $\frac{2\pi}{N}$.

- Corresponding convolution in time:

$$\tilde{x}[n] = x[n] * s[n] = \sum_{m=-\infty}^{\infty} x[n - mN]$$

  This shows that sampling in frequency gives rise to a periodic (aliased) sequence $\tilde{x}[n]$ in time.

Proof (4)

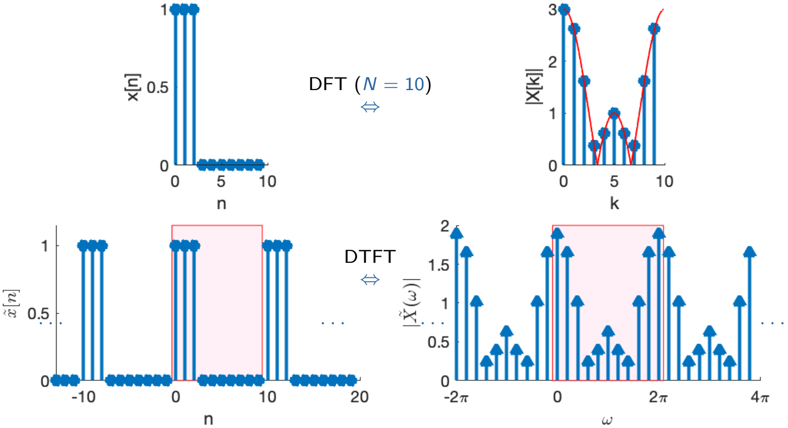
Finally, we relate $\tilde{x}[n]$ to the samples $X[k]$.

In terms of the $X[k]$, the I-DTFT gives

$$\begin{aligned}\tilde{x}[n] = \mathcal{F}^{-1}\left\{\tilde{X}(\omega)\right\} &= \frac{1}{2\pi}\int_0^{2\pi} \tilde{X}(\omega)e^{j\omega n}\mathrm{d}\omega \\ &= \frac{1}{2\pi}\frac{2\pi}{N}\int_0^{2\pi}\sum_{k=0}^{N-1} X[k]\delta(\omega - \tfrac{2\pi}{N}k)e^{j\omega n}\mathrm{d}\omega \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}kn}\end{aligned}$$

We have shown that the samples $X[k]$ correspond to $\tilde{x}[n]$, a periodic extension of $x[n]$. The relation is given by the DFT / IDFT equations.

DFT via DTFT



Properties of the DFT follow from those of the DTFT of their periodic extensions: $\tilde{x}[n] \Leftrightarrow \tilde{X}(\omega)$.

DFT – No aliasing

Assume just $L \leq N$ samples of $x[n]$ are nonzero, then

$$\tilde{x}[n] = x[n] \quad \text{for} \quad n = 0, \cdots, N-1$$

(no temporal aliasing). Then

$$\textbf{DFT:} \quad X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}, \qquad k = 0, \cdots, N-1$$

$$\textbf{IDFT:} \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn}, \qquad n = 0, \cdots, N-1$$

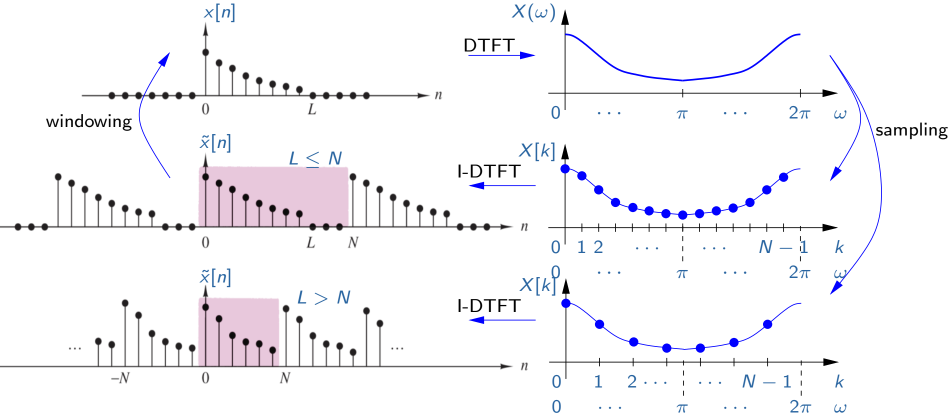
This is the usual definition of the DFT, but it is valid only under the assumption! The IDFT actually gives $\tilde{x}[n]$, a periodic sequence. Under the assumption, one period is equal to $x[n]$.

Since we recovered $x[n]$, we could reconstruct $X(\omega)$: the N samples $X[k]$ are sufficient to define $X(\omega)$.

DFT – Aliasing

If $L > N$, then the IDFT gives rise to temporal aliasing: the IDFT gives $\tilde{x}[n]$, but the central samples of $\tilde{x}[n]$ are not equal to $x[n]$.

In this case, the N samples $X[k]$ are **not** sufficient to reconstruct $X(\omega)$.



Reconstruction

Assume $N \geq L$: no aliasing. We can express the spectrum $X(\omega)$ in terms of the N samples $X[k] = X(\frac{2\pi}{N}k)$ using an interpolation formula:

$$X(\omega) = \sum_{n=0}^{N-1} x[n]e^{-j\omega n} \qquad \text{DTFT; now use } x[n] = \tilde{x}[n]\text{; insert IDFT}$$

$$= \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}kn} \right) e^{-j\omega n}$$

$$= \sum_{k=0}^{N-1} X[k] \left(\frac{1}{N} \sum_{n=0}^{N-1} e^{-j(\omega - \frac{2\pi}{N}k)n} \right)$$

Reconstruction

Since

$$\sum_{n=0}^{N-1} e^{-j\omega n} = \frac{1-e^{-j\omega N}}{1-e^{-j\omega}} = \frac{\sin(\frac{1}{2}\omega N)}{\sin(\frac{1}{2}\omega)} e^{-j\omega \frac{N-1}{2}} =: G(\omega)$$

we conclude that

$$X(\omega) = \frac{1}{N} \sum_{k=0}^{N-1} X[k] G\left(\omega - \tfrac{2\pi}{N} k\right)$$

$G(\omega)$ is the Dirichlet kernel, a "periodic sinc" function.

Reconstruction

Alternative derivation:

The original sequence $x[n]$ is obtained via windowing in time:

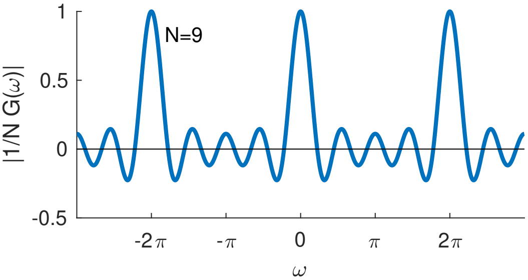
$$x[n] = \tilde{x}[n]\, g[n]\,, \qquad \text{where} \qquad g[n] = u[n] - u[n-N]$$

$g[n]$ is a pulse of length N which selects $x[n]$.

- The DTFT of $g[n]$ is $G(\omega)$ (Dirichlet) as defined before.
- A product in time gives convolution in frequency:

$$\begin{aligned} X(\omega) &= \frac{1}{2\pi}\tilde{X}(\omega) * G(\omega) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{2\pi}{N} \sum_k X[k]\delta(\theta - \tfrac{2\pi}{N}k) \right) G(\omega - \theta)\, \mathrm{d}\theta \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X[k] G(\omega - \tfrac{2\pi}{N}k) \end{aligned}$$

Reconstruction



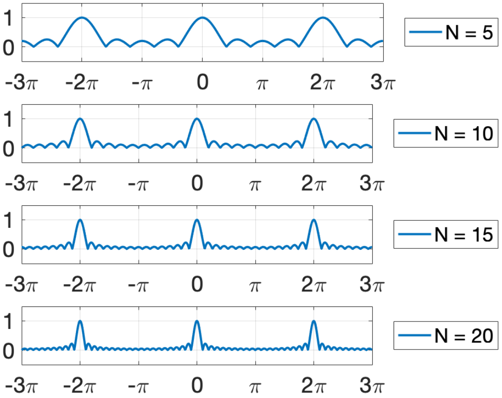
Clearly, $\frac{1}{N}G(\omega)$ is an interpolation function since

$$\frac{1}{N}G\left(\frac{2\pi}{N}k\right) = \begin{cases} 1, & k = 0 \\ 0, & k = 1, \cdots, N-1 \end{cases}$$

so that the interpolation formula gives $X(\frac{2\pi}{N}k) = X[k]$.

Reconstruction

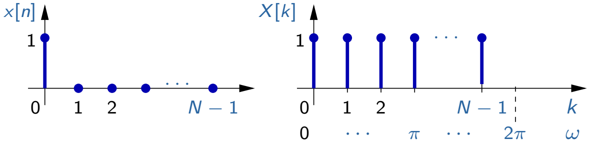
Plots of $|G(\omega)|$ for various N:



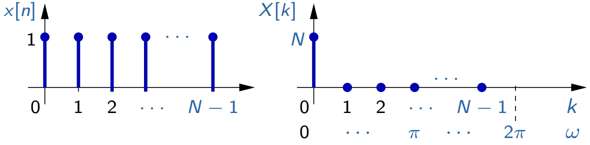
- The main lobe width is about $\frac{2\pi}{N}$. This determines the resolution in many applications.

DFT of basic signals

- Impulse: $\delta[n] \quad \Leftrightarrow \quad 1$



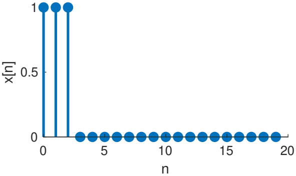
- Constant: $1 \quad \Leftrightarrow \quad N\delta[k]$

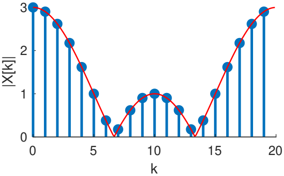


DFT of basic signals

- Pulse (length L): $g[n] \quad \Leftrightarrow \quad G[k]$

$$g[n] = u[n] - u[n-L]\,, \qquad G[k] = L\frac{\sin(\frac{\pi}{N}Lk)}{\sin(\frac{\pi}{N}k)}e^{-j\frac{\pi}{N}(L-1)}$$





(Note error in book wrt phase sign)

DFT of basic signals

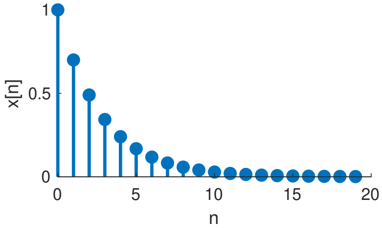
- (Clipped) exponential sequence:

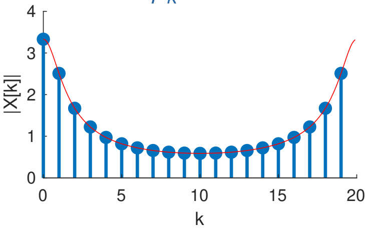
$$x[n] = a^n, \quad n = 0, \cdots, N-1 \Rightarrow X[k] = \sum_{n=0}^{N-1} a^n e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} \rho_k^n \quad \text{with } \rho_k = a\, e^{-j\frac{2\pi}{N}k}$$

$(\rho_k^N$ is due to clipping$)$

$$= \begin{cases} N & \text{if } \rho_k = 1 \\ \dfrac{1-\rho_k^N}{1-\rho_k} & \text{otherwise} \end{cases}$$

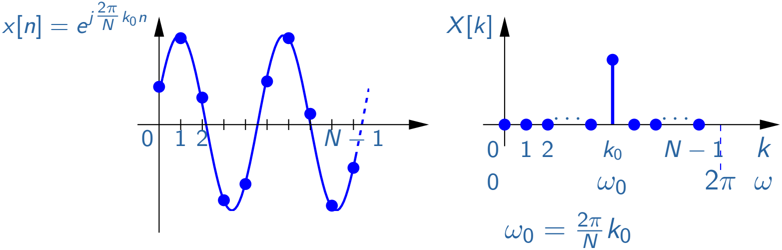




DFT of basic signals

- Complex exponential (exactly periodic: frequency multiple of $\frac{2\pi}{N}$)

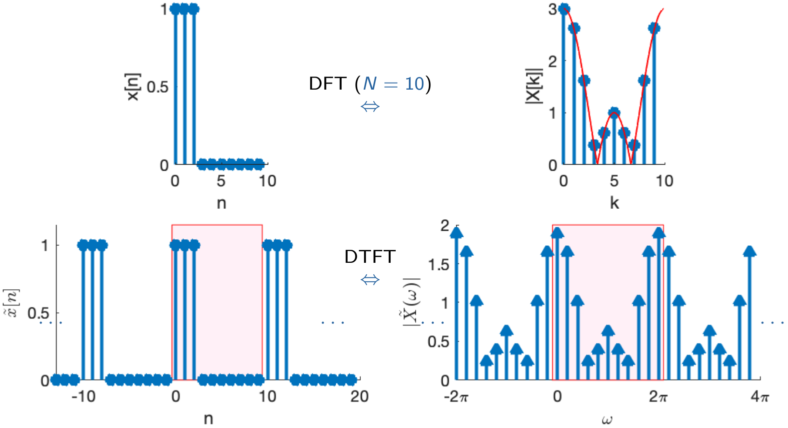
$$e^{j\frac{2\pi}{N}k_0 n} \quad \Leftrightarrow \quad N\,\delta[k-k_0 \mod N], \quad k = 0, \cdots, N-1$$



Properties

Let $x[n], \quad n = 0, \cdots, N-1 \quad \Leftrightarrow \quad X[k], \quad k = 0, \cdots, N-1$.

Underwater, the periodicity in time and frequency plays a role.



Circular time shift

Consider a delay (phase shift) of n_0 samples in frequency domain. On the extended sequence $\tilde{x}[n]$, we have

$$\text{DTFT:} \quad \tilde{x}[n - n_0] \quad \Leftrightarrow \quad \tilde{X}(\omega) e^{-j\omega n_0}$$

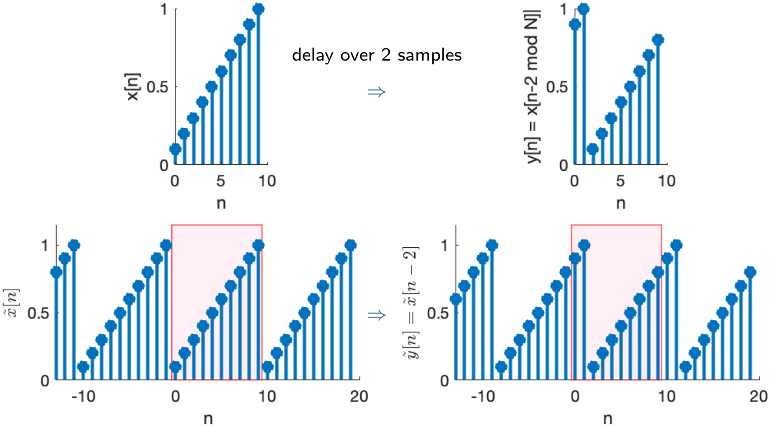
- Windowing $\tilde{x}[n - n_0]$ to the interval $0, \cdots, N-1$ gives

$$\text{DFT:} \quad x[n - n_0 \mod N] \quad \Leftrightarrow \quad X[k] e^{-j\frac{2\pi}{N} k n_0}$$

  In time, this is seen as a *circular* time shift.

Circular time shift

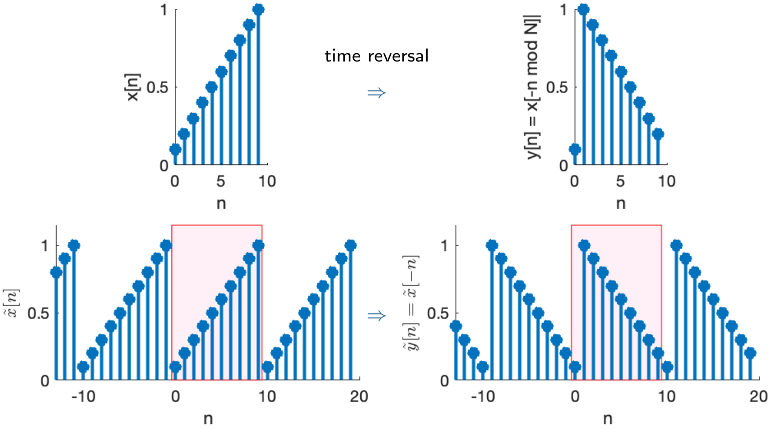
$x[n - n_0 \mod N]$ corresponds to a circular time shift over n_0 samples



Circular time reversal

For a real sequence:

$$x[-n \mod N] = \begin{cases} x[0] & n = 0 \\ x[N-n] & n = 1, \cdots, N-1 \end{cases} \quad \Leftrightarrow \quad X^*[k]$$



Complex conjugation

From the DTFT: $\tilde{x}^*[n] \Leftrightarrow \tilde{X}^*(-\omega)$, we find

$$x^*[n] \quad \Leftrightarrow \quad X^*[-k \mod N]$$

- Mapping $-k \mod N$ to the interval $0, \cdots, N$ gives

$$\text{DFT:} \quad x^*[n] \quad \Leftrightarrow \quad X^*[N-k]$$

  Hence, if $x[n]$ is real, then $X[k] = X^*[N-k]$, and

$$|X[k]| = |X^*[N-k]| \qquad \text{magnitude spectrum is even}$$
$$\angle X[k] = -\angle X[N-k] \qquad \text{phase spectrum is odd}$$
$$X[0] \text{ is real;}$$
$$X[N/2] \text{ is real for even } N$$

Circular convolution

For the DTFT, a (linear) convolution maps to a product in frequency

For the DFT, such a result holds for $\tilde{x}[n]$ and $\tilde{y}[n]$. This gives rise to a cyclic convolution:

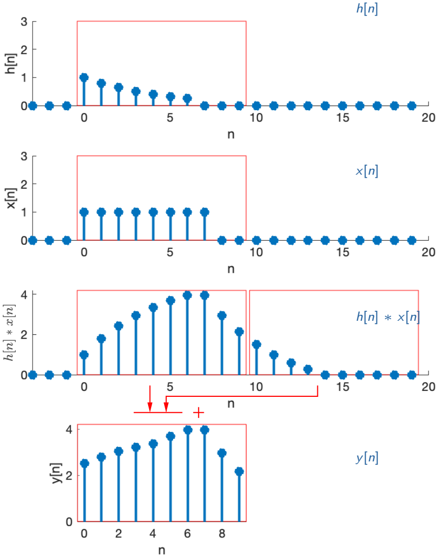
$$x[n] \circledast y[n] := \sum_{m=0}^{N-1} x[m] y[n-m \mod N] \quad \Leftrightarrow \quad X[k]\, Y[k]$$

Circular convolution

Compute a linear convolution $h[n] * x[n]$, then make periodic and window to 1 period.

$\Rightarrow$

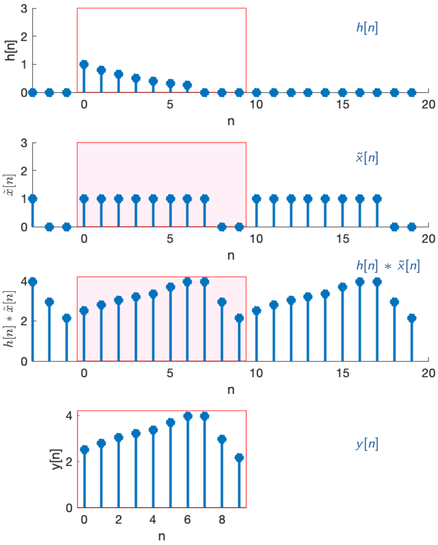
Becomes linear convolution **if** $N \geq N_x + N_h - 1$



Circular convolution

Alternative construction

Compute a linear convolution $h[n] * \tilde{x}[n]$ with periodic input, then window.



Circular convolution

$$x[n] \circledast y[n] \quad \Leftrightarrow \quad X[k]\,Y[k]$$

Proof: computing the IDFT of $X[k]Y[k]$ gives

$$\frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{m=0}^{N-1} x[m]e^{-j\frac{2\pi}{N}km}\right)\left(\sum_{\ell=0}^{N-1} y[\ell]e^{-j\frac{2\pi}{N}k\ell}\right)e^{j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} x[m]\sum_{l=0}^{N-1} y[\ell] \qquad \underbrace{\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(n-\ell-m)}}_{=\begin{cases} N, & n-\ell-m=0 \mod N \\ 0, & \text{otherwise}\end{cases}}$$

$$= \sum_{m=0}^{N-1} x[m]\,y[n-m \mod N]$$

Linear convolution implemented by cyclic convolution

- Circular convolution $y[n] = x[n] \circledast h[n]$ is equal to linear convolution $y[n] = x[n] * x[n]$ **if** $N \geq N_x + N_h - 1$.

- Implication: if the condition holds, we can implement linear convolution efficiently in frequency domain using $Y[k] = X[k]\, H[k]$.

  - Zero pad $x[n]$ and $h[n]$ to length $N = N_x + N_h - 1$
  - Compute the DFTs $X[k]$ and $H[k]$, $k = 0, \cdots, N-1$
  - Compute $Y[k] = X[k]\, H[k], \; k = 0, \cdots, N-1$
  - Compute the IDFT to obtain $y[n], \; n = 0, \cdots, N-1$.

- The DFTs are efficiently computed using the FFT [future lecture], with complexity $O(N \log_2 N)$.

  This has enabled the digital revolution with many applications that otherwise would not exist (jpg, mp3, wifi, radar, MRI, $\cdots$)

Multiplication

Multiplication in time $\Leftrightarrow$ circular convolution

$$x[n]\,y[n] \quad \Leftrightarrow \quad \frac{1}{N} X[k] \circledast Y[k]$$

Proof: dual of the proof for $x[n] \circledast y[n]$ (slide 31)

$$\mathcal{F}\{x[n]y[n]\} = \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{m=0}^{N-1} X[m] e^{j\frac{2\pi}{N}mn} \right) \left(\frac{1}{N} \sum_{\ell=0}^{N-1} Y[\ell] e^{j\frac{2\pi}{N}\ell n} \right) e^{-j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N^2} \sum_{m=0}^{N-1} X[m] \sum_{l=0}^{N-1} Y[\ell] \underbrace{\sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k-\ell-m)n}}_{= \begin{cases} N, & k-\ell-m=0 \mod N \\ 0, & \text{otherwise} \end{cases}}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} X[m]\, Y[k-m \mod N]$$

Circular frequency shift

Modulation is a spectial case of multiplication:

$$x[n]\, e^{j\frac{2\pi}{N}k_0 n} \quad \Leftrightarrow \quad X[k-k_0 \bmod N]$$

Proof: Previously, we saw

$$y[n] = e^{j\frac{2\pi}{N}k_0 n} \quad \Leftrightarrow \quad Y[k] = \delta[k-k_0 \bmod N]$$

Insert in the "multiplication" result: the DFT of $x[n]y[n]$ is

$$\begin{aligned}\frac{1}{N}X[k] \circledast Y[k] &= \frac{1}{N}\sum_{m=0}^{N-1} X[m]\, Y[k-m \bmod N] \\ &= \frac{1}{N}\sum_{m=0}^{N-1} X[m]\, N\delta[k-k_0-m \bmod N] = X[k-k_0 \bmod N]\end{aligned}$$

Energy (Parseval)

$$E_x = \sum_{n=0}^{N-1} |x[n]|^2 \quad = \quad \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

Proof:

$$\sum_{n=0}^{N-1} |x[n]|^2 = \sum_{n=0}^{N-1} x[n] x^*[n] = \sum_{n=0}^{N-1} x[n] \left(\frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn} \right)^*$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X^*[k] \left(\sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} \right)$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X^*[k] X[k] \quad = \quad \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

Summary

Table 10.1 on p.690:

Linearity	$ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
Complex conjugate	$x^*[n]$	$X^*[N-k \bmod N]$
Time shift	$x[n-n_0 \bmod N]$	$X[k]e^{-j\frac{2\pi}{N}kn_0}$
Time reverse	$x^*[-n \bmod N]$	$X^*[k]$
Frequency shift	$x[n]e^{j\frac{2\pi}{N}k_0 n}$	$X[k-k_0 \bmod N]$
Circ. convolution	$x[n] \circledast y[n]$ $= \sum_{m=0}^{N-1} x[m]y[n-m \bmod N]$	$X[k]\,Y[k]$
Multiplication	$x[n]\,y[n]$	$\frac{1}{N}X[k] \circledast Y[k]$ $= \frac{1}{N}\sum_{\ell=0}^{N-1} X[\ell]Y[k-\ell \bmod N]$
Parseval	$\sum_{n=0}^{N-1} \lvert x[n]\rvert^2$	$\frac{1}{N}\sum_{k=0}^{N-1} \lvert X[k]\rvert^2$
Symmetry	$x[n]$ real	$X[k] = X[N-k \bmod N]$ $\lvert X[k]\rvert = \lvert X[N-k \bmod N]\rvert$ $\angle(X[k]) = -\angle(X[N-k \bmod N])$

To do:

- Study chapter 10
- Try to make exercise ...

Next lecture, we consider the construction of spectra (chapter 14). We revisit the DFT later, when we look at the FFT (chapter 11).