EE3S1 Signal Processing – DSP

Lecture 2: Sampling (Ch. 6.1–6.4)

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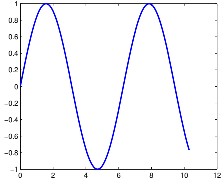
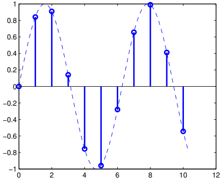
8 October 2025

Sampling – revisited

$x(t)$ 

$\Rightarrow$

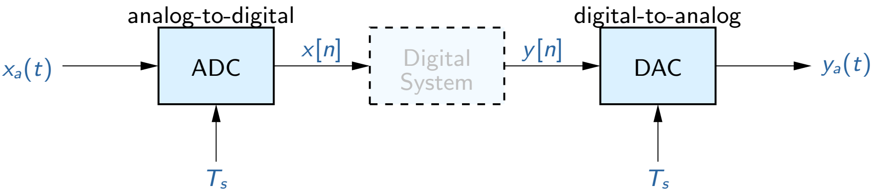
$x[n]$ 

- Recap of ideal sampling and reconstruction
- Bandpass sampling
- Nonideal reconstruction

Prior knowledge: EE2S1 Signals & Systems

- Continuous-time Fourier transform
- Sampling and reconstruction (refreshed today)

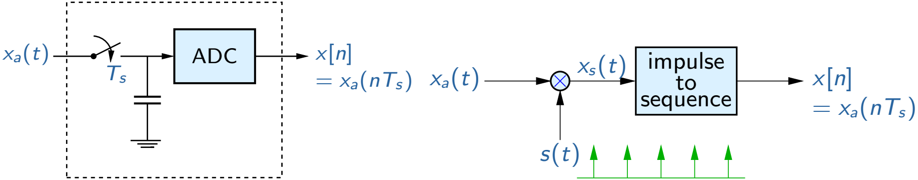
Ideal sampling



In absence of a digital system ($y[n] = x[n]$), the goal for ideal sampling/reconstruction is to have $y_a(t)$ equal to $x_a(t)$.

- We write $x_a(t)$ rather than $x(t)$ so that we can better distinguish the spectra of $x_a(t)$ and $x[n]$ in the notation

Ideal sampling



T_s is the sampling period, $F_s = \frac{1}{T_s}$ is the sampling frequency [Hz].

- We first model "ideal sampling", where the sampled signal is represented by a train of delta pulses $x_s(t)$

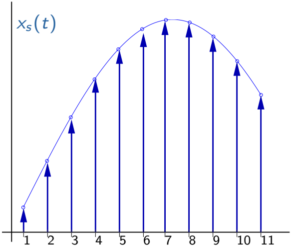
Ideal sampling

Using

$$s(t) = \sum_n \delta(t - nT_s) \qquad \text{impulse train}$$

we can write

$$x_s(t) = x_a(t)\, s(t) = \sum_n x(nT_s)\delta(t - nT_s) = \sum_n x[n]\delta(t - nT_s)$$



- $x_s(t)$ and $x[n]$ have a 1-to-1 relation (same information content) — we first analyze $x_s(t)$.

Ideal sampling

The Fourier transform of $x_s(t)$ is

$$X_s(\Omega) = \mathcal{F}\left\{\sum_n x(nT_s)\delta(t - nT_s)\right\} = \sum_n x_a(nT_s)\mathcal{F}\left\{\delta(t - nT_s)\right\}$$
$$= \sum_n x_a(nT_s)e^{-j\Omega T_s n} \qquad = \sum_n x[n]e^{-j\omega n}$$

For a sequence $x[n]$, the Discrete-Time Fourier Transform (DTFT) is therefore defined as

$$X(\omega) = \sum x[n]e^{-j\omega n}$$

so that $X(\omega) = X_s(\Omega)$, where $\omega = \Omega T_s$ is the normalized angular frequency.

- $X(\omega)$ is periodic with period 2π: it suffices to consider a single period: $-\pi \le \omega \le \pi$, the *fundamental interval*.

Relation of $X_s(\Omega)$ to $X_a(\Omega)$

- The Fourier transform of $s(t)$ is also a series of delta pulses:

$$S(\Omega) = \frac{2\pi}{T_s} \sum_k \delta(\Omega - k\Omega_s)\,, \qquad \Omega_s = \frac{2\pi}{T_s} = 2\pi F_s$$

- The Fourier transform of $x_s(t) = x_a(t)\, s(t)$ is then

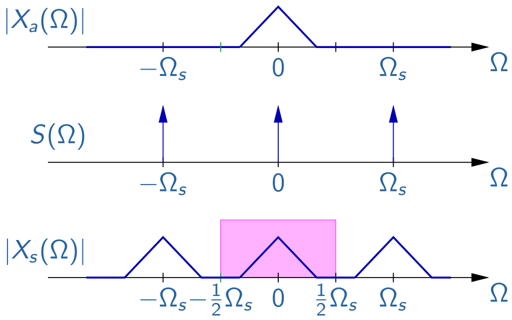
$$X_s(\Omega) = \frac{1}{2\pi} X_a(\Omega) * S(\Omega) = \frac{1}{T_s} \sum_k X_a(\Omega - k\Omega_s)$$

This is a sum of shifted spectra of $X_a(\Omega)$.

Aliasing

The spectrum $X_s(\Omega)$ is periodic with period Ω_s.

- The fundamental interval is $-\frac{1}{2}\Omega_s \leq \Omega \leq \frac{1}{2}\Omega_s$, which corresponds to $-\pi \leq \omega \leq \pi$.
- The summation leads to *aliasing*.

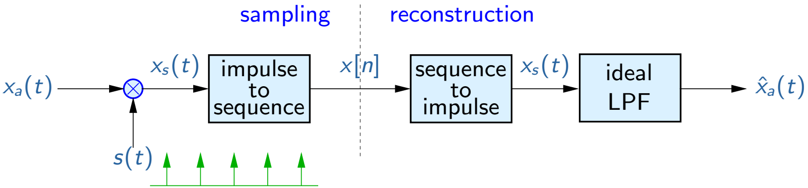


Nyquist condition

We do not want the copies $X_a(\Omega - k\Omega_s)$ to overlap each other.

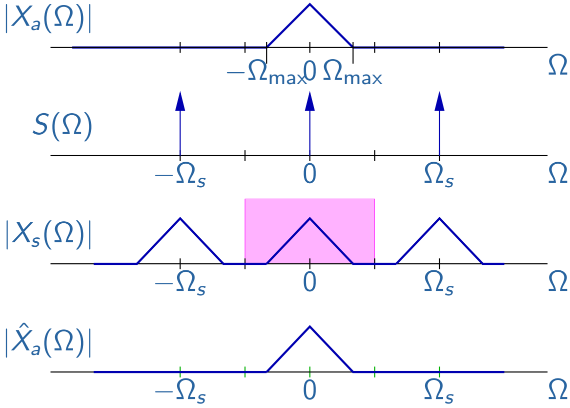
The **Nyquist condition** says that no overlap occurs if $x(t)$ is bandlimited with maximal frequency $\Omega_{\max} < \frac{1}{2}\Omega_s$; and in that case $x_a(t)$ can be recovered from its samples $x[n]$ (*Shannon theorem*).

- This minimal sampling frequency $2\Omega_{\max}$ is the *Nyquist frequency*.
- Indeed, in that case we can recover $x_a(t)$ from $x_s(t)$ using an ideal lowpass filter with passband $\Omega_c = \frac{1}{2}\Omega_s$.



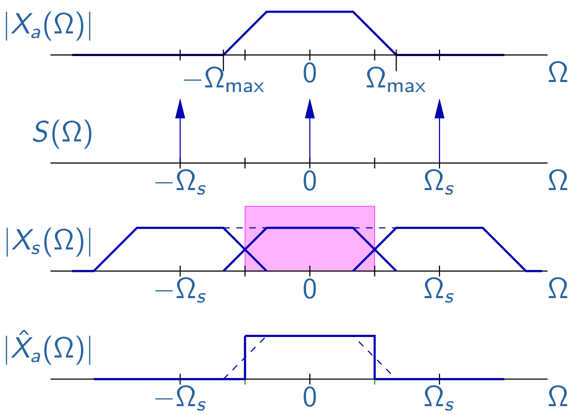
Nyquist condition satisfied

If $\Omega_{\max} < \frac{1}{2}\Omega_s$, then the shifted spectra don't overlap, and perfect reconstruction is possible.



Nyquist condition not satisfied

If $\Omega_{\max} \geq \frac{1}{2}\Omega_s$, then the shifted spectra overlap, and perfect reconstruction is not possible (destructive aliasing)



Ideal reconstruction

If the Nyquist condition is satisfied, we can recover $x(t)$ from $x_s(t)$ using an ideal lowpass filter ("brick-wall filter"):

$$X_a(\Omega) = X_s(\Omega) H_r(\Omega) \qquad \Leftrightarrow \qquad \begin{aligned} x_a(t) &= x_s(t) * h_r(t) \\ &= \sum x[n] h_r(t - nT_s) \end{aligned}$$

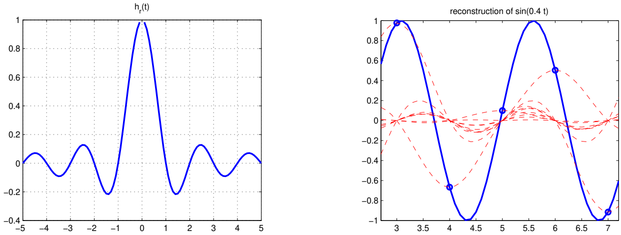
where

$$H_r(\Omega) = \begin{cases} T_s & |\Omega| < \frac{1}{2}\Omega_s \\ 0 & \text{otherwise} \end{cases} \qquad \Leftrightarrow \qquad h_r(t) = \frac{\sin(\pi t / T_s)}{\pi t / T_s} = \text{sinc}(\pi \frac{t}{T_s})$$

- The filter is called an anti-imaging filter, interpolation filter, or reconstruction filter. The reconstruction is a sum of scaled and shifted sinc functions.
  (Note: "sinc" has various definitions, with or without π)

Ideal reconstruction

$$\hat{x}_a(t) = \sum x[n] h_r(t - nT_s)$$



Note that $h_r(mT_s) = \begin{cases} 1 & m = 0 \\ 0 & \text{otherwise} \end{cases}$

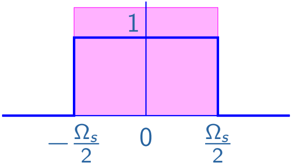
$$\Rightarrow \quad \hat{x}_a(mT_s) \; = \; \sum_n x[n]\, h_r((m-n)T_s)) \; = \; x[m]$$

Hence, the filter smoothly *interpolates* the $x[m]$.

Non-ideal sampling: Anti-aliasing filter

The anti-aliasing filter is an analog filter which we apply before the ADC, to remove all frequencies above $\frac{1}{2}\Omega_s$.

$$H_{aa}(\Omega) = \begin{cases} 1, & |\Omega| \leq \frac{1}{2}\Omega_s \\ 0, & \text{elders} \end{cases}$$



This prevents aliasing later during sampling. (The distortion of frequencies above $\frac{1}{2}\Omega_s$ due to this filter is unavoidable.)

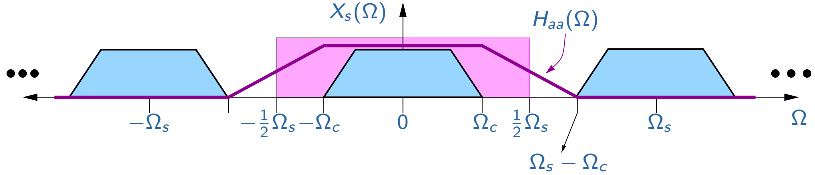
- A potential problem is that a sharp filter (small transition band) has high complexity, while a not-so-sharp filter might still give aliasing.
- A sharp IIR filter might also give phase distortion at high frequencies.

Two solutions follow.

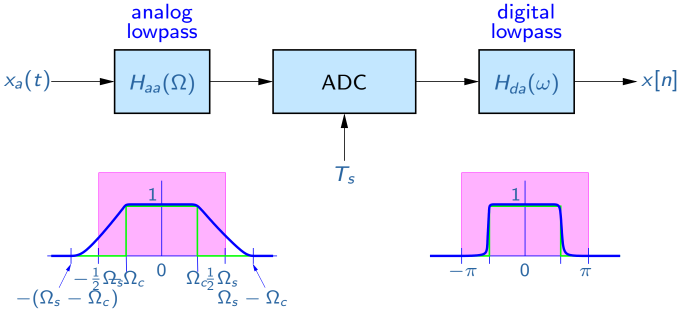
Anti-aliasing filter
Approach 1

- Choose the desired cut-off frequency Ω_c.
  E.g., for audio, set $\Omega_c = 2\pi \cdot 20$ kHz, the Nyquist rate will be $F_s = 40$ kHz
- Take a slightly higher sampling rate: $\Omega_s = (1 + r)2\Omega_c$, with $0 < r < 1$, e.g., $\Omega_s = 2\pi \cdot 44$ kHz
- We can now use a non-ideal anti-aliasing filter $H_{aa}(\Omega)$, with a transition band between Ω_c and $\frac{1}{2}\Omega_s$.
  In fact, the stopband can be even a bit larger: $\Omega_s - \Omega_c$.

Anti-aliasing filter



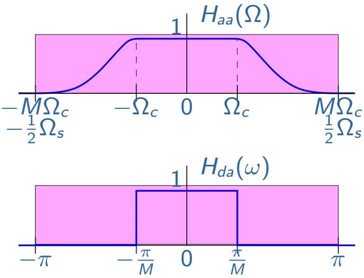
After sampling, we can digitally filter out the undesired components between Ω_c and $\frac{1}{2}\Omega_s$.



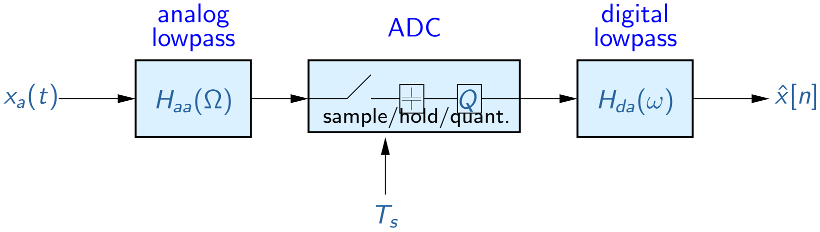
Anti-aliasing filter

Approach 2 [details in next lecture]

- Use a cheap $H_{aa}(\Omega)$ with a wide transition band
- Oversample $x_a(t)$ with a large factor M, e.g., $\Omega_s = 8\Omega_c = 2M\Omega_c$
- After sampling, digitally remove the unwanted frequencies above Ω_c, and reduce the sample rate by M. [Covered in a future lecture.]



Practical ADC



The hold circuit is often based on a capacitance (buffer) to maintain the voltage. The switch is often a MOSFET transistor.

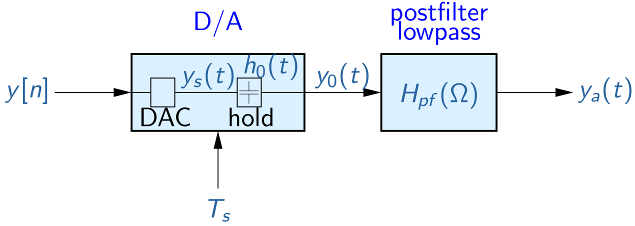
The quantizer requires a voltage reference, a voltage divider, and a series of comparators.

Nonideal reconstruction

Ideal reconstruction with a sinc filter is not implementable:

- The delta impulses in $y_s(t)$ are not realizable
- The sinc filter is not causal
- The impulse response has infinite duration

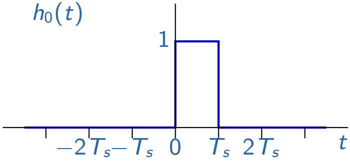
Instead, we can use other filters such as a zero-order hold or a first-order hold

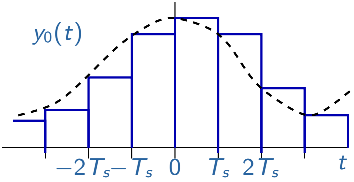


Zero-order hold

The zero-order hold circuit replaces the sequence of $x[n]$ by a step-wise changing analog signal:

$$h_0(t) = \begin{cases} 1, & 0 \le t \le T_s \\ 0, & \text{otherwise} \end{cases} \qquad \begin{aligned} y_0(t) &= y_s(t) * h_0(t) \\ &= \sum y[n] h_0(t - nT_s) \end{aligned}$$

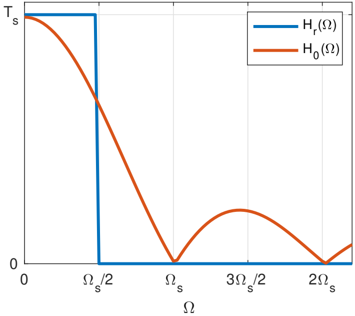




An analog postfilter (lowpass filter) smooths the step signal and removes remaining high frequencies

Zero-order hold – frequency domain analysis

$$Y_0(\Omega) = Y_s(\Omega) H_0(\Omega) \,, \qquad H_0(\Omega) = T_s \, \mathrm{sinc}(\pi \frac{\Omega}{\Omega_s}) \, e^{-j\pi \frac{\Omega}{\Omega_s}}$$

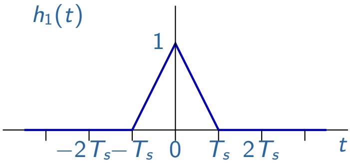


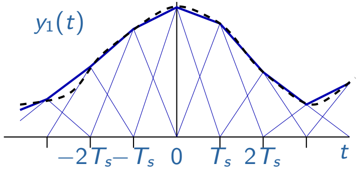
$H_0(\Omega)$ is not a sharp filter and aliased components around Ω_s, $2\Omega_s$ will remain. The postfilter has to remove these.

Linear interpolation [not in book]

Linear interpolation ("first order interpolation") is obtained by convolving $y_s(t)$ with a tent shape:

$$h_1(t) = \begin{cases} t, & -T_s \le t \le 0 \\ T_s - t, & 0 \le t \le T_s \\ 0, & \text{otherwise} \end{cases} \qquad y_1(t) = \sum y[n] h_1(t - nT_s)$$

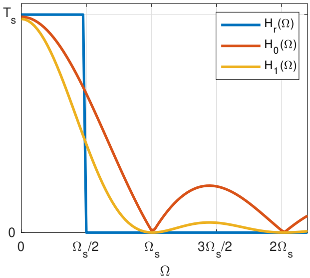




Linear interpolation — frequency domain

Recall that the convolution of a rectangular pulse with itself gives a tent shape. Hence

$$h_1(t) = \frac{1}{T_s}\, h_0(t) * h_0(-t) \quad \Leftrightarrow \quad H_1(\Omega) = T_s[\mathrm{sinc}(\pi\frac{\Omega}{\Omega_s})]^2$$

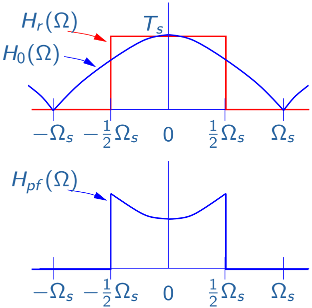


Due to the square, the linear interpolation filter will give much more suppression around Ω_s and $2\Omega_s$.

Nonideal reconstruction

In general, $H_0(\Omega) \neq H_r(\Omega)$: no ideal interpolation. A postfilter can correct for this:

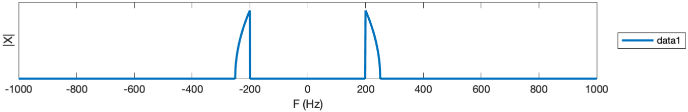
$$H_{pf}(\Omega) = \frac{H_r(\Omega)}{H_0(\Omega)} = \begin{cases} T_s/H_0(\Omega) & |\Omega| < \frac{1}{2}\Omega_s \\ 0 & \text{otherwise} \end{cases}$$



- The shape correction in the passband could also be implemented in digital domain.

Bandpass sampling [not in book]

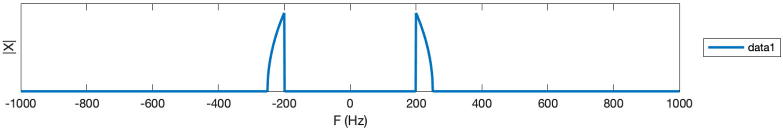
A bandpass signal with bandwidth B and center frequency F_c is a signal with nonzero spectral content at frequencies F defined by $0 < F_L < |F| < F_H$, where $F_c = \frac{1}{2}(F_L + F_H)$ and $B = F_H - F_L$.



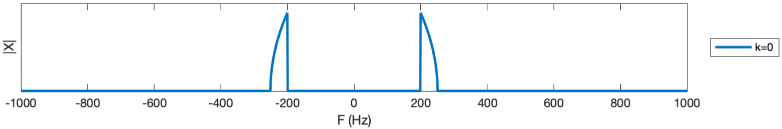
$F_L = 200$ Hz, $F_H = 250$ Hz, $B = 50$ Hz, $F_c = 225$ Hz

According to the sampling theory, we should sample with $F_s = 500$ Hz.

Nyquist sampling of a bandpass signal

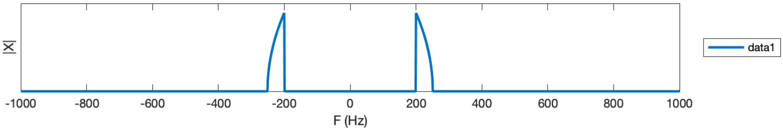


$$\Downarrow \quad X(F) = F_s \sum_{k=-\infty}^{\infty} X_a(F - kF_s) \text{ with } F_s = 500 \text{ Hz}$$



Most of the band is empty: not very efficient to sample at 500 Hz.

Nyquist sampling of a bandpass signal



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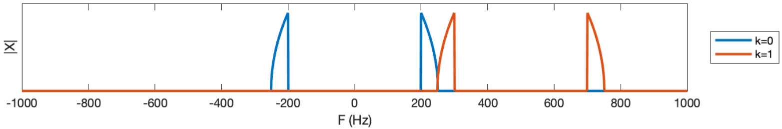

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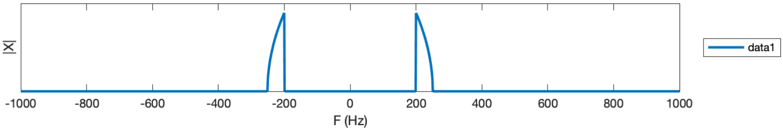

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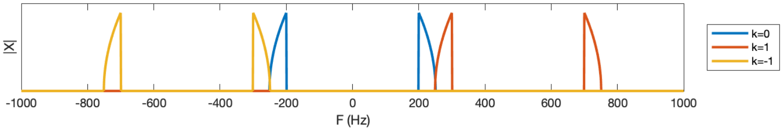

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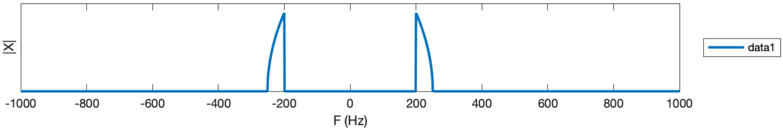
Nyquist sampling of a bandpass signal



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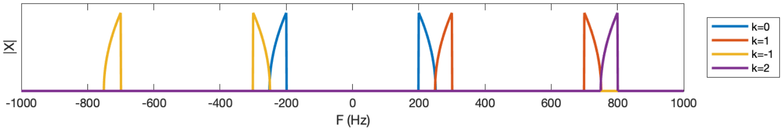

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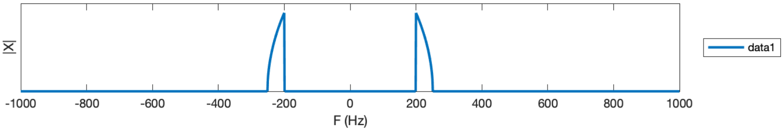
Nyquist sampling of a bandpass signal



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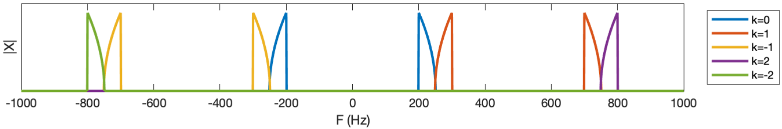
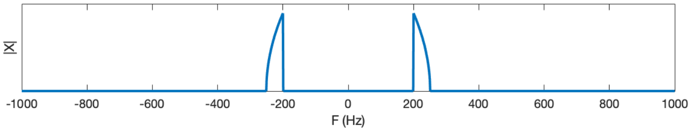


Most of the band is empty: not very efficient to sample at 500 Hz.

Sampling of a bandpass signal: example 1

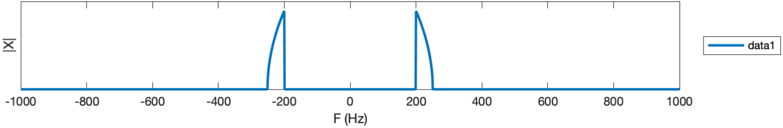
Integer band positioning

If $F_H = mB$, sampling with $F_s = 2B$ is possible without destructive aliasing.

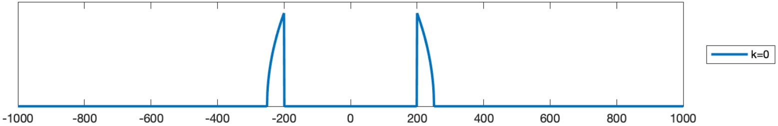


In the current example $F_H = 250$, $B = 50$ and $m = 5$.

Sampling of a bandpass signal: example 1

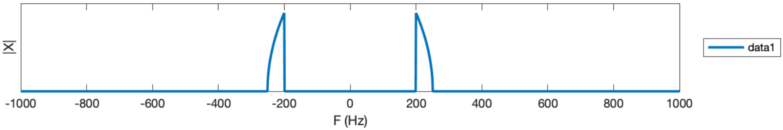


$$\Downarrow X(F) = F_s \sum_{k=-\infty}^{\infty} X_a(F - kF_s) \text{ with } F_s = 2B = 100 \text{ Hz}$$



$m = 5$ is odd; the band $F \in \{F_L \cdots F_H\}$ ends up at $F \in \{0 \cdots B\}$

Sampling of a bandpass signal: example 1



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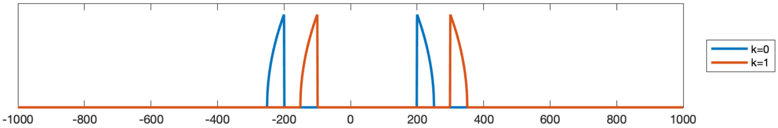

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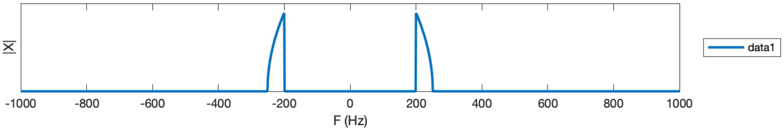
Sampling of a bandpass signal: example 1



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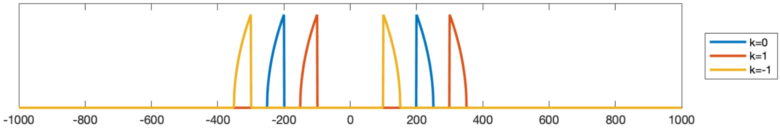

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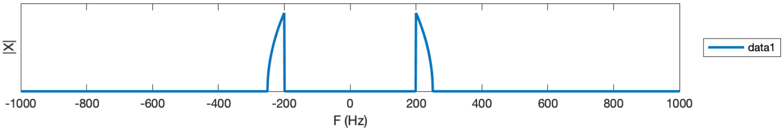
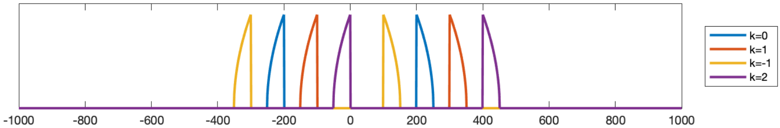
Sampling of a bandpass signal: example 1



$$\Downarrow X(F) = F_s \sum_{k=-\infty}^{\infty} X_a(F - kF_s) \text{ with } F_s = 2B = 100 \text{ Hz}$$



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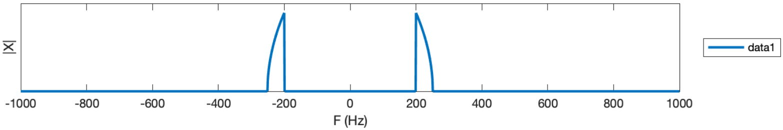

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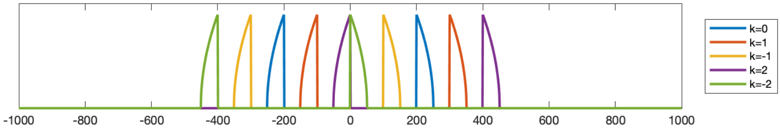

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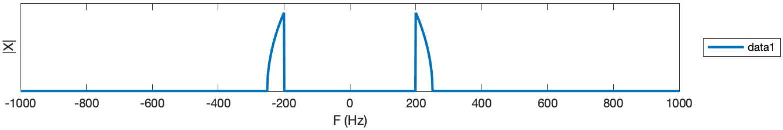

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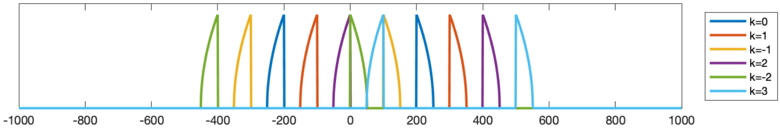

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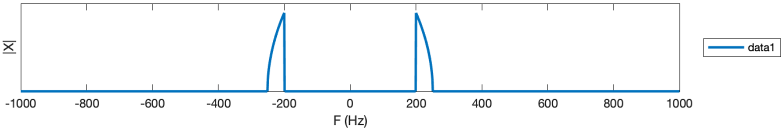

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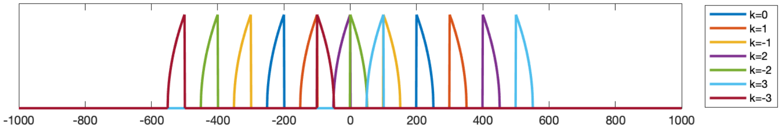

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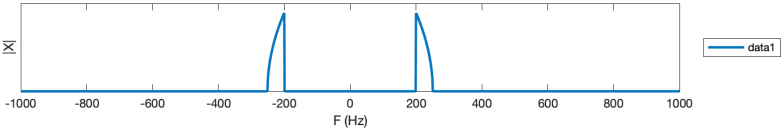

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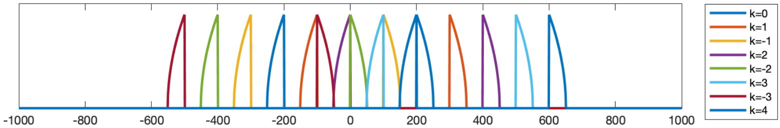

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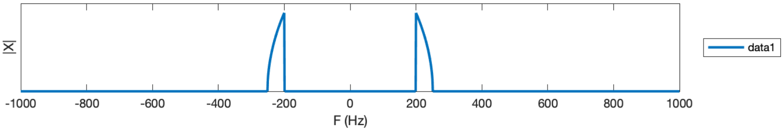

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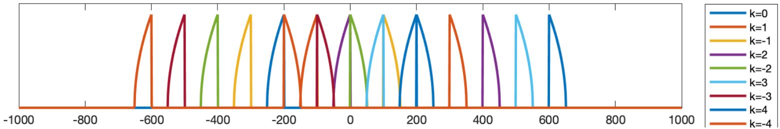

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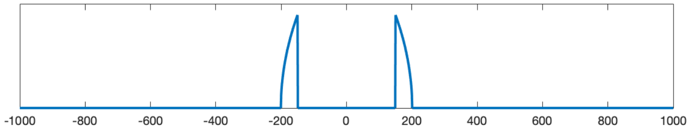
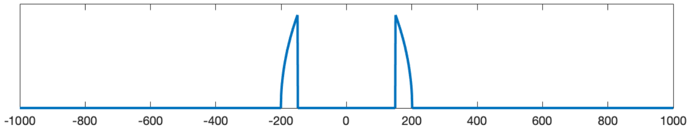
Sampling of a bandpass signal: example 2



In the second example $F_H = 200$, $B = 50$ and $m = 4$.

$m = 4$ is even; the band $F \in \{F_L \cdots F_H\}$ ends up at $F \in \{-B \cdots 0\}$: *inverted*!

Sampling of a bandpass signal: example 2



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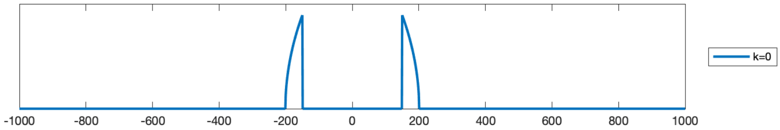

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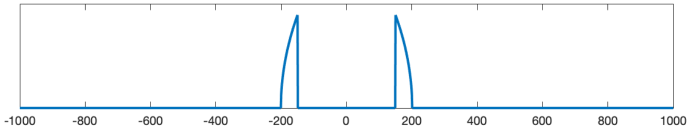

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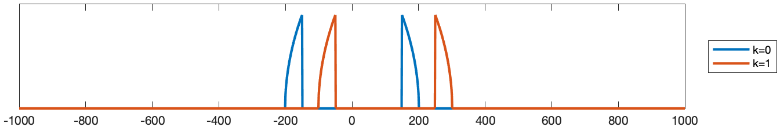

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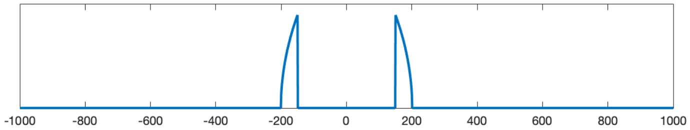

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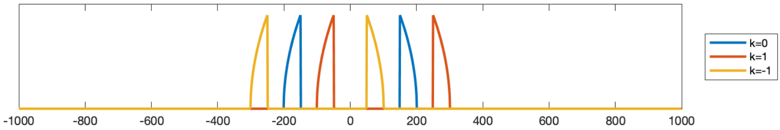

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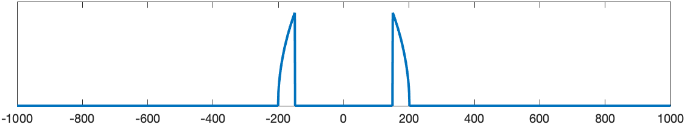

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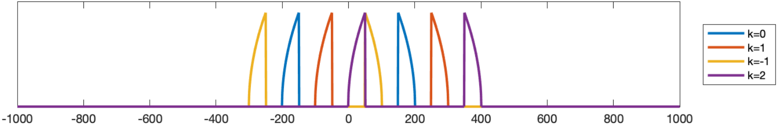

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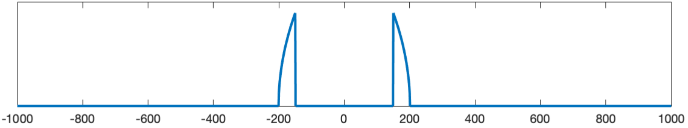

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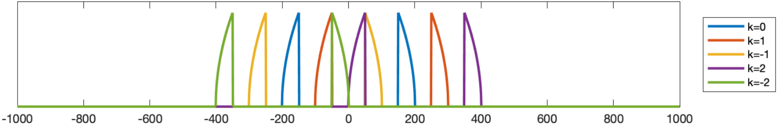

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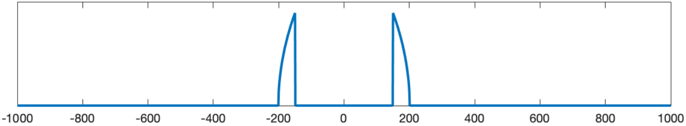

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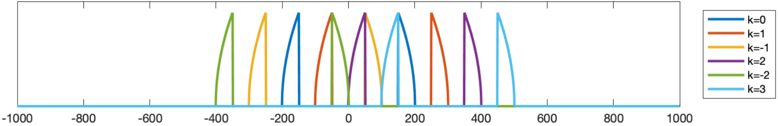

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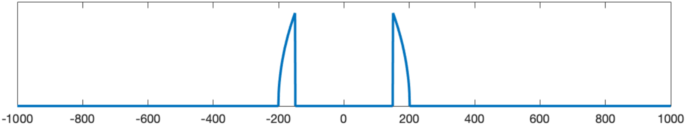

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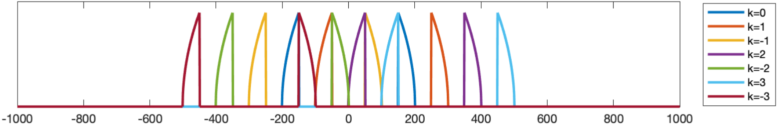

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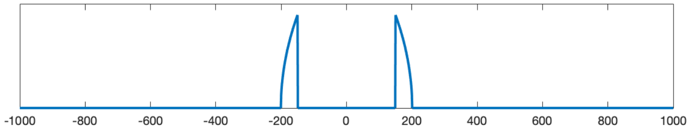

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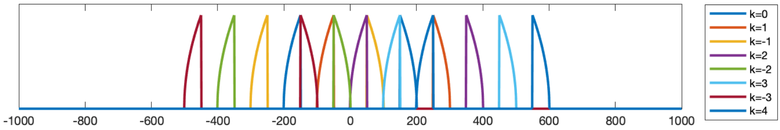

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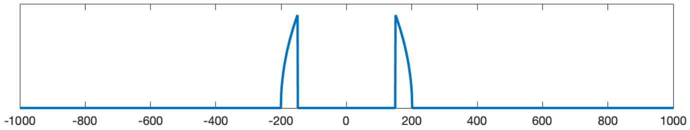

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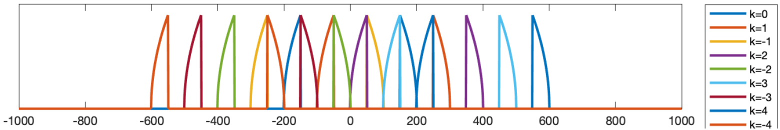

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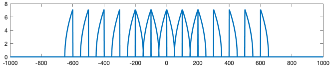
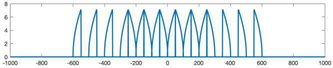
Reconstruction and downconversion

m odd



The spectra of the sampled even and odd band positioned signals are both free from aliasing

m even



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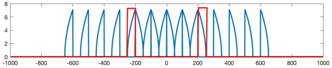

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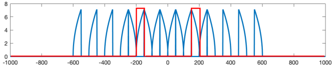


The original signal can be reconstructed using a bandpass filter $g(t)$:

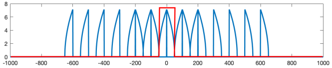
$$x_a(t) = \sum_{n=-\infty}^{\infty} x[n] g(t - nT), \text{ with}$$

$$g(t) = \frac{\sin \pi B t}{\pi B t} \cos 2\pi F_c t$$

Note: $g(t)$ is equal to the interpolation function of bandlimited signals $h_r(t)$, modulated with the carrier frequency F_c

Reconstruction and downconversion

m odd



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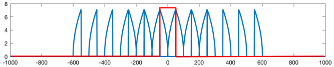


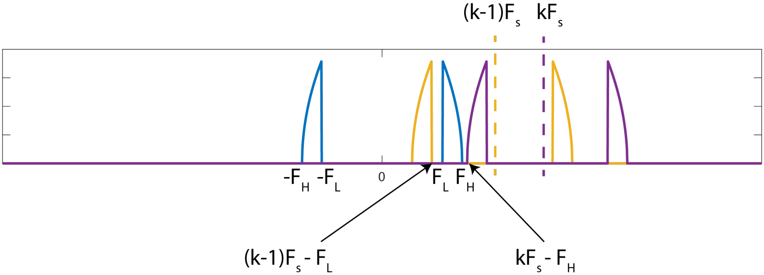
Downconversion to baseband: alternatively, we may reconstruct the signal at baseband

The baseband spectra with $m =$ even are inverted.
This is easily corrected in digital domain: take $y[n] = (-1)^n x[n]$.

Arbitrary band positioning

Now consider $F_H \neq mB$. How to choose F_s?

Spectra are shifted by kF_s and we need to avoid aliasing!

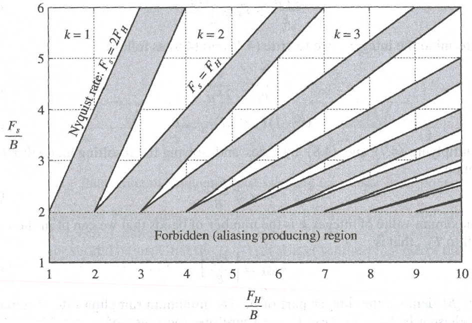


Conditions for F_s:

$$\begin{cases} (k-1)F_s - F_L & \leq & F_L \\ kF_s - F_H & \geq & F_H \end{cases} \implies \frac{2}{k}\left(\frac{F_H}{B}\right) \leq \frac{F_s}{B} \leq \frac{2}{k-1}\left(\frac{F_H}{B} - 1\right)$$

Arbitrary band positioning

How to choose F_s?



For our signal, we know F_H and B.

Then, we can choose an F_s/B along the vertical line corresponding to F_H/B

(This is rather detailed and you are not expected to be able to reproduce this.)

To do:

- Refresh your memory of EE2S1 Signals & Systems: sampling theory. It is summarized in Chapter 6.1–6.4.
- Study new material:
  - Anti-aliasing filter
  - Non-ideal reconstruction
  - Bandpass sampling
- Try to make exercise ...

Next lecture, we consider upsampling and downsampling (Ch. 6.5).