EE3S1 Signal Processing – DSP

Lecture 1: Introduction and Recap (Ch. 2–5)

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Course topics – DSP part

- Sampling - revisited
- Downsampling and upsampling   courselab 1
- Discrete Fourier Transform (DFT)
- Spectral analysis   courselab 2
- Sigma-Delta ADC   courselab 3
- Fast Fourier Transform (FFT)

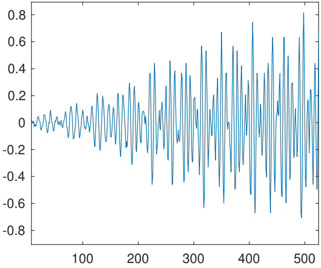
Prior knowledge: EE2S1 Signals & Systems (refreshed today)

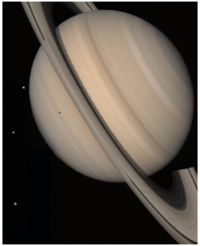
- Discrete-time signals, LTI systems, convolution
- DTFT, z-transform
- Sampling and reconstruction

Introduction

What is a signal?

- 1-D: Speech, communication signal (output of an antenna): $x[n]$
- 2-D: Image: $s[i,j]$
- 3-D: Video: $s[i,j,n]$
- N-D: Output of N antennas (stacked in a vector): $\boldsymbol{x}[n]$

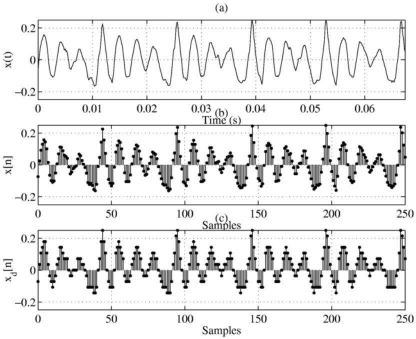


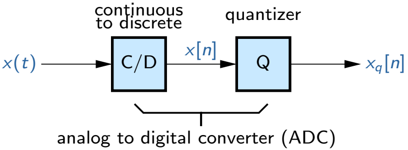


Introduction

Classes of signals

- Continuous time (analog) vs. discrete time (digital)
- Continuous amplitude vs. quantized
- Deterministic (DSP) vs. random (SSP)

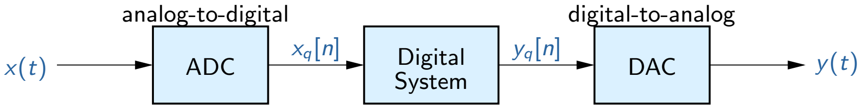




Introduction

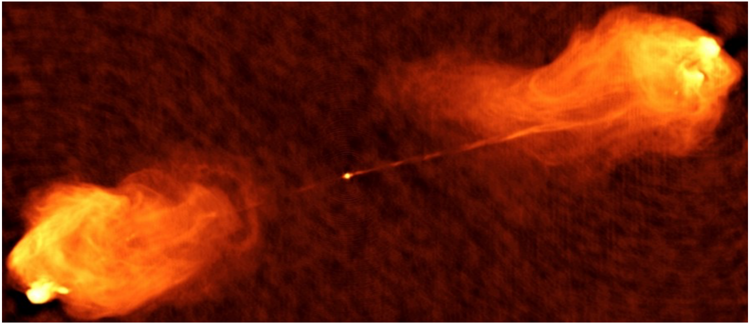
Goals of signal processing

- Processing of analog signals using discrete-time operations / digital hardware
- Estimation of parameters (properties of the signal)
- Analysis of the system that is in between an input and an output signal (the "channel")
- Modeling of such signals/systems (cf. machine learning)



Example application: radio astronomy

Cygnus A - a quasar



Model: a large collection of point sources; the q-th source $s_q(t)$ at location (pixel) $\mathbf{z}_q$ has variance (power) σ_q^2, which is shown in the intensity image.

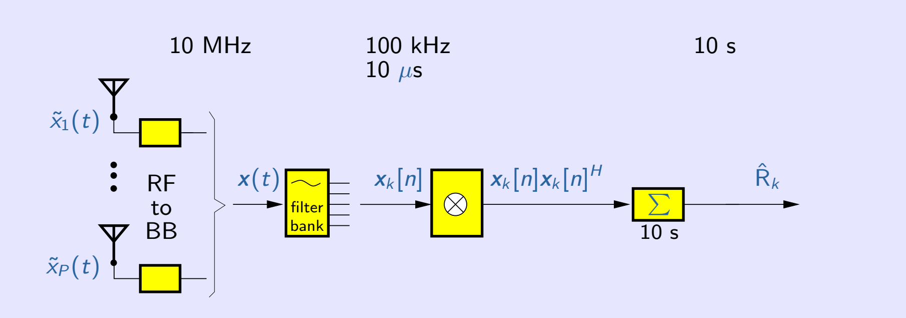
Example application: radio astronomy

The Very Large Array (VLA) - New Mexico

The antenna signals from the 27 dishses are stacked in a vector $\tilde{\boldsymbol{x}}(t)$.

Example application: radio astronomy

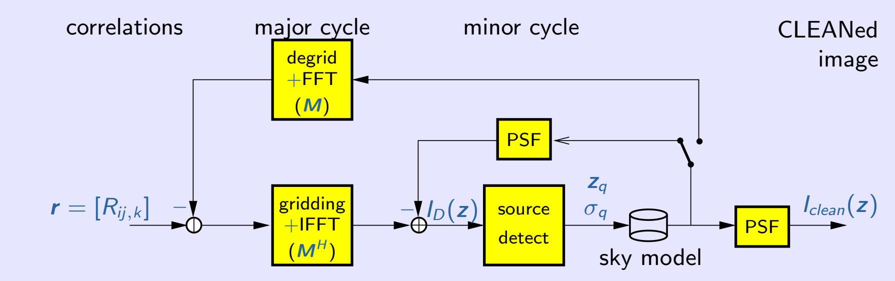
Data processing



The noisy signals are moved to baseband, split into small time-frequency bins, and correlated to each other to form short-term correlation matrix estimates $\hat{\boldsymbol{R}}_k$. These are stored.

Example application: radio astronomy

Image formation



The observed correlations are stacked in a vector $\boldsymbol{r}$ (a few million entries). To form the image, the observation matrix $\boldsymbol{M}$ has to be inverted; this numerically tricky step is done iteratively using FFTs. In the minor cycle, detected sources are subtracted from the data.

Recap: Discrete-time signals - Ch.1

A discrete-time signal is an infinite sequence

$$x[n], \; n = \cdots, -1, 0, 1, 2, \cdots$$

We write

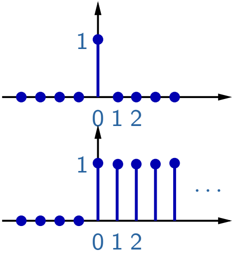
$$x = \left[\cdots, x[-1], \boxed{x[0]}, x[1], x[2], \cdots\right]$$

where the box denotes time 0.

- We usually refer to the signal als $x[n]$, but correct is simply x.
- The square brackets denote that the indices n are integers

Recap: Discrete-time signals - Ch.1

Basic signals



- Impulse $\delta[n] = [\cdots, 0, \boxed{1}, 0, 0, \cdots]$
  (does not have infinite amplitude!)
- Step $u[n] = [\cdots, 0, \boxed{1}, 1, 1, \cdots]$
- Pulse of width N: $p[n] = u[n] - u[n-N]$
- Complex exponential sequence:
  $x[n] = A\,\alpha^n u[n]$, with $A, \alpha \in \mathbb{C}$

We can write a signal as a sum of (scaled) shifted impulses:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

Energy and power

- The energy of a signal $x[n]$ is

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

- The power is the average energy per sample:

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

- ℓ_2 is the space of signals with finite energy. More in general:

$$\ell_p = \left\{ x \mid \left(\sum |x[n]|^p\right)^{1/p} < \infty \right\}$$

Systems

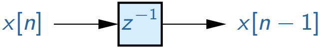
A system $\mathcal{T}$ transforms a signal x into a signal y.

$$y = \mathcal{T}\{x\}$$

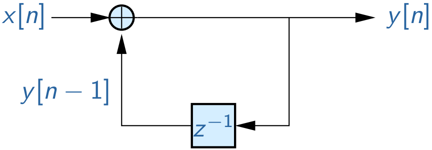
We often write with abuse of notation $y[n] = \mathcal{T}\{x[n]\}$

Examples

- Time shift (delay):
  $y[n] = x[n-1]$



- Reverse: $y[n] = x[-n]$
- Moving-average (MA):
  $y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$
- Summer (accumulator):
  $$y[n] = \sum_{k=-\infty}^{n} x[k] = x[n] + y[n-1]$$



Systems

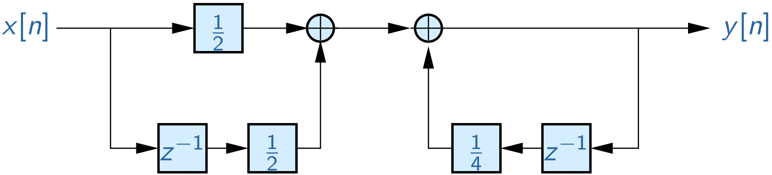
- Some systems can be described by a Linear Constant Coefficient Differential Equation:

$$y[n] + a_1 y[n-1] + a_2 y[n-2] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

- The output signal can be computed via a recursion:

$$y[n] = -a_1 y[n-1] - a_2 y[n-2] + b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

$$y[n] = \frac{1}{4} y[n-1] + \frac{1}{2} x[n] + \frac{1}{2} x[n-1]$$



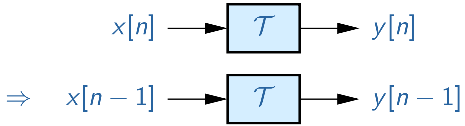
Systems

- A system is linear if it satisfies superposition:

$$\begin{aligned} y_1[n] &= \mathcal{T}\{x_1[n]\} \\ y_2[n] &= \mathcal{T}\{x_2[n]\} \end{aligned} \quad \Rightarrow \quad \alpha_1 y_1[n] + \alpha_2 y_2[n] = \mathcal{T}\{\alpha_1 x_1[n] + \alpha_2 x_2[n]\}$$

- A system is time-invariant if a delayed input leads to a delayed output:

$$y[n] = \mathcal{T}\{x[n]\} \quad \Rightarrow \quad y[n-1] = \mathcal{T}\{x[n-1]\}$$



In this course, we will mostly (but not always) consider LTI systems.

Systems

- A system is causal if the output at time n does not depend on future values of the input.
- A signal is bounded if it has a maximum amplitude x_{max}:

$$|x[n]| < x_{\text{max}}$$

  The signal is in ℓ_∞.
- A system is "bounded-input bounded output" (BIBO) stable if any bounded input leads to a bounded output: $\mathcal{T} : \ell_\infty \to \ell_\infty$.

Recap: Convolution and the impulse response - Ch.2

- The impulse response of a system is the response to an impulse at time 0:

$$h[n] = \mathcal{T}\{\delta[n]\}$$

- A finite impulse response (FIR) system has $h[n]$ of finite length, otherwise it is an infinite impulse response (IIR) system.

- An LTI system is completely described by its impulse response.

Convolution and the impulse response - Ch.2

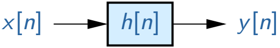
Proof:

Recall $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$.

$$y[n] = \mathcal{T}\{x[n]\} = \mathcal{T}\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\} = \sum_{k=-\infty}^{\infty} x[k]\mathcal{T}\{\delta[n-k]\}$$

$$= \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

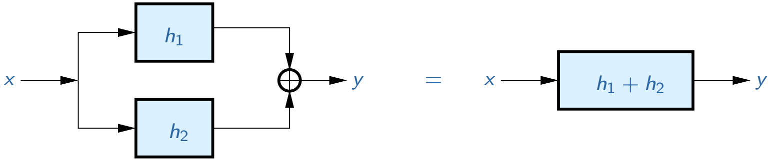
- This is the convolution sum. Notation: $yn] = x[n] * h[n]$.



Properties of the convolution

Convolution has similar properties as a multiplication (but acts on sequences):

- Commutative: $x * h = h * x \quad \Rightarrow$ cascade/series: $h_1 * h_2 = h_2 * h_1$
- Associative: $(x * g) * h = x * (g * h)$
- Distributive: $x * (h_1 + h_2) = x * h_1 + x * h_2 \quad \Rightarrow$ parallel



Properties of the convolution

- The "unit element" of the convolution is $\delta[n]$:

$$x[n] * \delta[n] = x[n]$$

- A unit delay has impulse response $\delta[n-1]$:

$$x[n] * \delta[n-1] = x[n-1]$$

Properties of the convolution

- A causal system has $h[n] = 0$ for $n < 0$ (no response before the impulse arrives).
- A BIBO stable system has an impulse response that is absolutely summable:

$$\sum_{-\infty}^{\infty} |h[n]| < \infty$$

  That is: $h[n] \in \ell_1$.

- An FIR system is always stable.

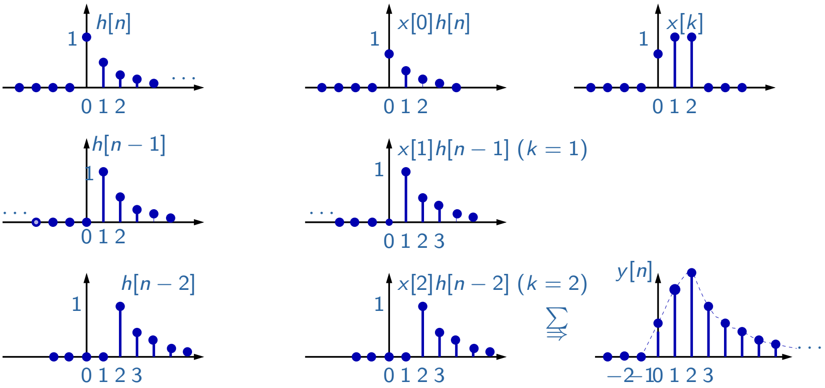
Convolution as a matrix-vector product

$$\begin{bmatrix} \vdots \\ y[-2] \\ y[-1] \\ \boxed{y[0]} \\ y[1] \\ y[2] \\ \vdots \end{bmatrix} = \begin{bmatrix} \ddots & & & & & & \\ \cdots & h[0] & & & & 0 & \\ \cdots & h[1] & h[0] & & & & \\ \cdots & h[2] & h[1] & \boxed{h[0]} & & & \\ \cdots & h[3] & h[2] & h[1] & h[0] & & \\ \cdots & h[4] & h[3] & h[2] & h[1] & h[0] & \\ & \vdots & \vdots & \vdots & \vdots & & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ x[-2] \\ x[-1] \\ \boxed{x[0]} \\ x[1] \\ x[2] \\ \vdots \end{bmatrix}$$

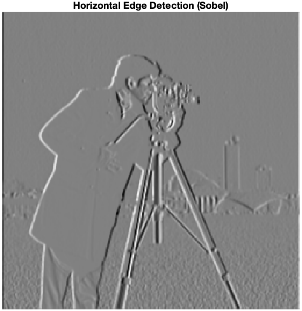
- linear $\leftrightarrow$ matrix-vector;
- causal $\leftrightarrow$ lower triangular
- time invariant $\leftrightarrow$ constant along diagonals ("Toeplitz")

Computing the convolution

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \; = \; \cdots + x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + \cdots$$



Convolution in 2D



In 2D convolution, each pixel is replaced by a weighted sum of its neighbor pixels (the kernel specifies the weights)

Convolution kernel ("Sobel-x"): $h[i,j] = \begin{bmatrix} -1 & 0 & 1 \\ -2 & \boxed{0} & 2 \\ -1 & 0 & 1 \end{bmatrix}$

Recap: The z-transform - Ch. 4

The z-transform of a discrete-time signal $x[n]$ is defined as

$$X(z) = \mathcal{Z}(x[n]) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- For example:

$$x = [\cdots, 0, 1, 2, \boxed{3}, 4, 5, 0, \cdots] \Rightarrow X(z) = z^2 + 2z^1 + 3 + 4z^{-1} + 5z^{-2}$$

$$x[n] = a^n u[n] \Rightarrow$$

The z-transform

Properties:

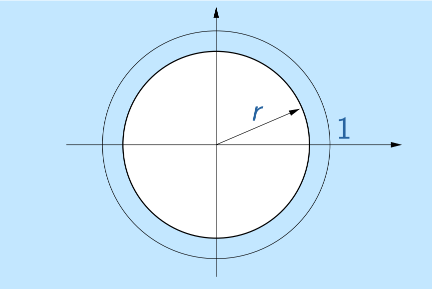
$$
\begin{array}{lcll}
ax[n] + by[n] & \Leftrightarrow & aX(z) + bY(z) & \\
x[n-k] & \Leftrightarrow & z^{-k}X(z) & \\
a^n x[n] & \Leftrightarrow & X(\frac{z}{a}) & \text{often } a = e^{j\omega_c} \\
x[-n] & \Leftrightarrow & X(z^{-1}) & \\
x[n] = \delta[n] & \Leftrightarrow & X(z) = 1 &
\end{array}
$$

The z-transform

Convergence

Along with $X(z)$, we should specify the region of convergence (ROC).

- Different $x[n]$ can give the same $X(z)$ but with different ROC.
- Generally we are only interested in ROCs that contain the unit circle (where the Fourier transform will be defined).



The z-transform

The transfer function

- An LTI system is defined by its impulse response $h[n]$. Its z-transform is

  $$H(z) = \sum h[n] z^{-n}$$

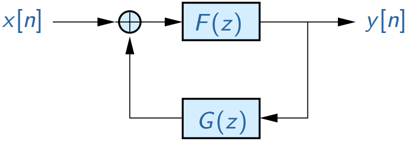
  It is called the transfer function.

- The output of the system is the convolution sum $y[n] = x[n] * h[n]$. Its z-transform is

  $$\begin{aligned} Y(z) &= \sum_n \left\{ \sum_k x[k] h[n-k] \right\} z^{-n} \\ &= \sum_k x[k] \left\{ \sum_n h[n-k] z^{-(n-k)} \right\} z^{-k} \\ &= X(z) H(z) \end{aligned}$$

The z-transform

Analyzing systems with feedback loops



$$Y(z) = F(z)\big(X(z) + G(z)Y(z)\big)$$
$$Y(z)(1 - F(z)G(z)) = F(z)X(z)$$
$$Y(z) = \frac{F(z)}{1 - F(z)G(z)}X(z) \qquad \Leftrightarrow \qquad H(z) = \frac{\text{direct path}}{1 - \text{loop}}$$

- Feedback loops result in rational transfer functions
- But: the derivation is also valid if F and/or G are rational transfer functions themselves!

The z-transform
Rational transfer function

- For the LCCDE

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k],$$

  we find after the z-transform

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\Leftrightarrow \; H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} =: \frac{B(z)}{A(z)}$$

- Thus, the transfer function is a rational function.

The z-transform

Poles and zeros

Consider the rational transfer function

$$H(z) = \frac{B(z)}{A(z)}$$

- The poles of $H(z)$ are the solutions of $A(z) = 0$, the zeros are the solutions of $B(z) = 0$.

  Some of them could cancel each other.
- We also have to consider poles and zeros at $z = 0$ and $z = \infty$.

  If we do that, the number of poles is equal to the number of zeros.
- By definition, the ROC cannot contain any poles.

The z-transform

- **Causal system** For a causal LTI system, we have $h[n] = 0, \; n < 0$.

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n} = h[0] + h[1]z^{-1} + \cdots$$

  Consequently, an LTI system is causal iff the ROC includes the outsize of a circle, including $z \to \infty$

- **Stable system** A system is BIBO stable iff $\displaystyle\sum_{n=-\infty}^{\infty} |h[n]| < \infty$.

  Note that

$$|H(z)| \;\leq\; \sum |h[n]z^{-n}| \;=\; \sum |h[n]| \, |z^{-n}|$$

  Therefore, $|H(e^{j\omega})| < \infty$: the unit circle is in the ROC.

- A causal stable system has all poles within the unit circle.

Recap: The discrete-time Fourier Transform - Ch. 3

Definition

$$X(\omega) = \mathcal{F}\{x[n]\} := \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- $X(\omega + 2\pi) = X(\omega)$: periodic in ω, period 2π:
  It suffices to consider the interval $\omega \in [-\pi, \pi]$ (the fundamental interval)
- $X(\omega) = |X(\omega)|e^{j\phi(\omega)}$,
  where $|X(\omega)|$: amplitude spectrum, $\phi(\omega)$: phase spectrum

The Discrete-Time Fourier Transform

Relation to the z-transform

Given $X(z)$, the DTFT is obtained by taking $z = e^{j\omega}$

It is necessary that $|z| = 1$ is in the ROC.

Many books write the DTFT as $X(e^{j\omega})$.

Consequently,

- $Y(\omega) = H(\omega)X(\omega)$
- $H(\omega) = \sum h[n]e^{-j\omega n}$

The Discrete-Time Fourier Transform

Inverse DTFT

$$x[n] \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n} \mathrm{d}\omega \;=\; \int_{-1/2}^{1/2} X(f) e^{j2\pi f n} \mathrm{d}f \quad (\omega = 2\pi f)$$

Energy and Parseval

$$E_x \;=\; \sum_{n=-\infty}^{\infty} |x[n]|^2 \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 \mathrm{d}\omega$$

$S_x(\omega) := |X(\omega)|^2$ is called the energy spectral density

The Discrete-Time Fourier Transform

Real-valued signals

If $x[n] = x^*[n]$, then

$$X^*(\omega) = X(-\omega) \qquad = X(z^{-1})\big|_{z=e^{j\omega}}$$

and hence we have symmetry relations:

$$|X(-\omega)| = |X(\omega)|, \qquad \phi(-\omega) = -\phi(\omega)$$

In this case, it suffices to plot the spectrum on $0 \leq \omega \leq \pi$

The Discrete-Time Fourier Transform

Properties

$$
\begin{array}{lcl}
ax[n] + by[n] & \Leftrightarrow & aX(\omega) + bY(\omega) \\
x[n-k] & \Leftrightarrow & e^{-j\omega k} X(\omega) \\
x[-n] & \Leftrightarrow & X(-\omega) \\
x^*[n] & \Leftrightarrow & X^*(-\omega) \\
 & & \\
(x_1 * x_2)[n] & \Leftrightarrow & X_1(\omega) X_2(\omega) \\
x[n]y[n] & \Leftrightarrow & (X * Y)(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\lambda) Y(\omega - \lambda) \mathrm{d}\lambda \\
 & & \\
e^{j\omega_0 n} x[n] & \Leftrightarrow & X(\omega - \omega_0) \\
x[n] \cos \omega_0 n & \Leftrightarrow & \frac{1}{2}(X(\omega - \omega_0) + X(\omega + \omega_0))
\end{array}
$$

The Discrete-Time Fourier Transform

Example

Consider

$$x[n] = a^n u[n]\,, \qquad -1 < a < 1$$

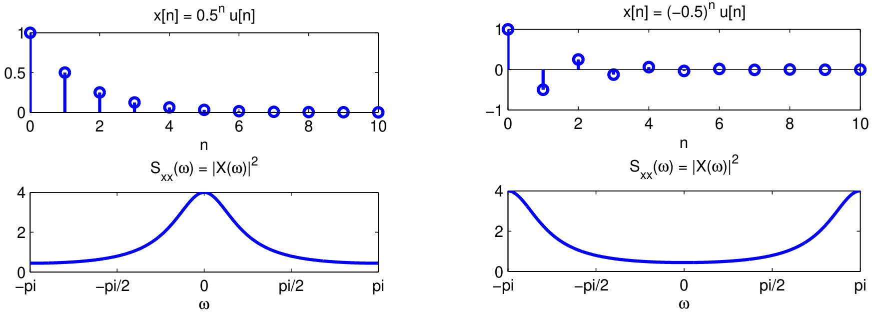
We find

$$X(\omega) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$

$$\begin{aligned} S_x(\omega) = |X(\omega)|^2 = X(\omega)X^*(\omega) &= \frac{1}{(1 - ae^{-j\omega})(1 - ae^{j\omega})} \\ &= \frac{1}{1 - 2a\cos\omega + a^2} \end{aligned}$$

The Discrete-Time Fourier Transform

Example



The Discrete-time Fourier Transform
Example (2)

Consider

$$x[n] = \begin{cases} A, & 0 \leq n \leq L-1 \\ 0, & \text{elders} \end{cases}$$

We find

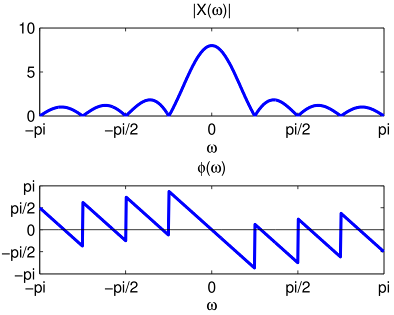
$$X(\omega) = \sum_{n=0}^{L-1} A e^{-j\omega n} = A \frac{1 - e^{-j\omega L}}{1 - e^{-j\omega}} = A e^{-j(\omega/2)(L-1)} \frac{\sin(\omega L/2)}{\sin(\omega/2)}$$

This function is called the Dirichlet kernel: a "periodic sinc"

The Discrete-time Fourier Transform

Example (2)

$$|X(\omega)| = |A| \, \left| \frac{\sin(\omega L/2)}{\sin(\omega/2)} \right| , \qquad X(0) = |A| L$$



$(L = 8)$

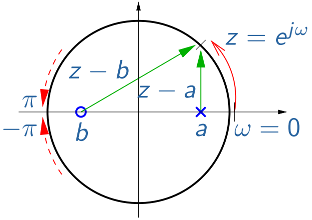
zero crossings for
$\omega = \pm \frac{2\pi}{L} k \; (k \neq 0)$

phase slope $\leftrightarrow$ delay
phase jumps (π) $\leftrightarrow$ sign changes

Phasors

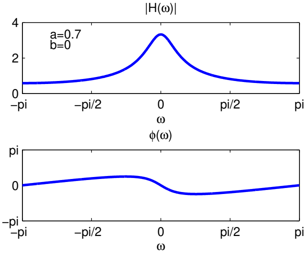
Consider a rational transfer function $H(z) = \dfrac{z-b}{z-a}$.

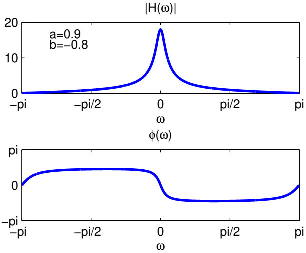
How to draw $|H(\omega)|$ and $\phi(\omega)$?



$$|H(\omega)| = \frac{|z-b|}{|z-a|}$$

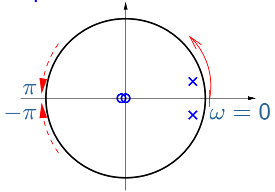
$$\phi(\omega) = \angle(z-b) - \angle(z-a) \mod 2\pi$$

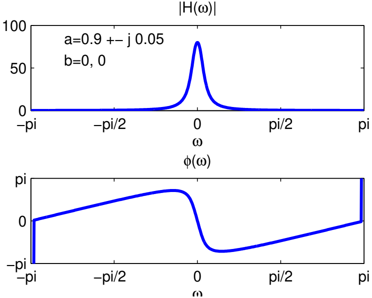




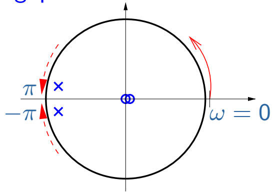
Phasors

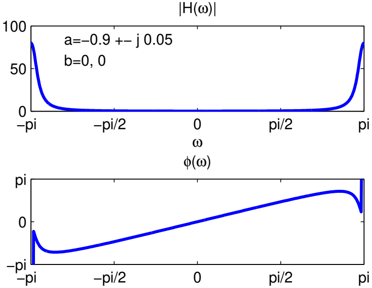
Lowpass





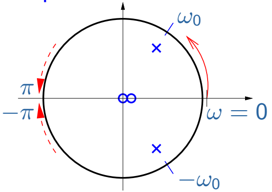
Highpass

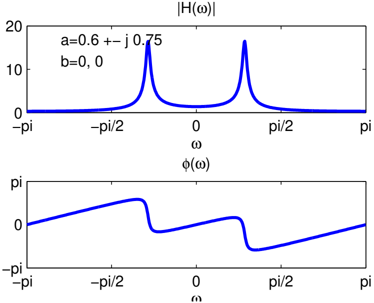




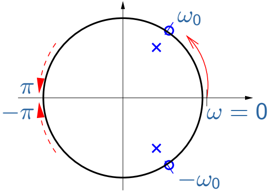
Phasors

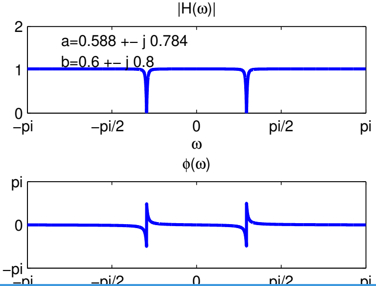
Bandpass





Notch





Linear phase filters

An LTI system has linear phase if its frequency response can be written as

$$H(\omega) = A(\omega)e^{-j(\alpha\omega - \beta)}$$

in which $A(\omega)$ and α, β are real-valued.

We have $A(\omega) = \pm|H(\omega)|$

- $e^{-j\alpha\omega}$ is interpreted as the response of a delay, $z^{-\alpha}$.
  If $|H(\omega)|$ is flat in its passband (ideal filter), then it does not distort the shape of an input signal in its passband (e.g., a pulse)
- More generally, consider the group delay $\tau_g = -\dfrac{\mathrm{d}\phi(\omega)}{\mathrm{d}\omega}$
  For a bandpass signal, this specifies the delay of its envelope.

Linear phase filters

A filter can have linear phase only if it is FIR with order M, of the form

$$h[n] = h[M-n] \quad \text{(symmetric)}$$

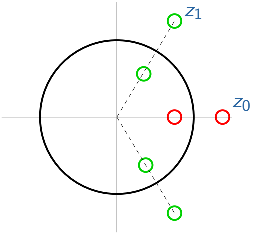
or

$$h[n] = -h[M-n] \quad \text{(antisymmetric)}$$

Location of zeros:

From $h[n] = \epsilon h[M-n]$, $\epsilon = \pm 1$, it follows $H(z) = \epsilon z^{-M} H(z^{-1})$.

- If z_0 is a zero, then also $1/z_0$ is a zero.
- If $h[n]$ is also real-valued, then also z_0^* and $1/z_0^*$ are zeros.



Allpass filters - Ch. 5.7 [New]

$H(z)$ is an allpass filter if $|H(e^{j\omega})| = 1$ for all ω.

Example: $H(z) = z^{-1}$.

- Every rational allpass function with real-valued coefficients has the form

$$H(z) = \frac{a_M + a_{M-1}z^{-1} + \cdots + a_1 z^{-M+1} + z^{-M}}{1 + a_1 z^{-1} + \cdots + a_{M-1}z^{-M+1} + a_M z^{-M}} = \frac{z^{-M}A(z^{-1})}{A(z)}$$

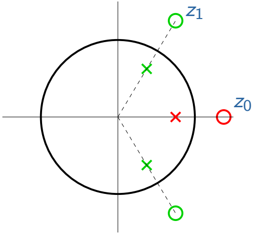
  Hence, the numerator polynomial is the reverse of the denominator. (For complex coefficients, also a conjugation is needed.)

- Proof:

$$|H(e^{j\omega})| = \frac{|A(e^{-j\omega})|}{|A(e^{j\omega})|} = \frac{|A^*(e^{j\omega})|}{|A^(e^{j\omega})|} = 1$$

Allpass filter

If z_0 is a zero of $H(z)$, then $1/z_0$ is a pole. Hence, complex poles and zeros come in groups of 4: z_0^* is also a zero, and $1/z_0^*$ is a pole.



For a stable causal allpass filter, all zeros are outside the unit circle.

Minimum phase filters

$H(z)$ is a minimum phase filter if all poles and zeros are within the unit circle.

Hence, also $H^{-1}(z)$ is causally stable.

- In contrast, an allpass filter has all zeros outside the unit circle; its inverse is causally unstable but anti-causally stable.

Minimum phase filters

A causal stable transfer function can be factorized as

$$H(z) = H_{mf}(z)H_{ap}(z), \text{ with } H_{mf}(z) \text{ minimum phase, } H_{ap}(z) \text{ allpass}$$

The algorithm for this is:

- Assign all poles of $H(z)$ to $H_{mf}(z)$.
- Assign all zeros of $H(z)$ located within the unit circle to $H_{mf}(z)$, and those outside the unit circle to $H_{ap}(z)$.
- The zeros of $H_{ap}(z)$ also specify the poles of $H_{ap}(z)$.
- These in turn are assigned as zeros to $H_{mf}(z)$

Example:

$$H(z) = \frac{1 - 3z^{-1}}{1 - 0.5z^{-1}} = \frac{1 - 1/3\, z^{-1}}{1 - 0.5z^{-1}} \cdot \frac{1 - 3z^{-1}}{1 - 1/3\, z^{-1}}$$

To do:

- Refresh your memory of EE2S1 Signals & Systems: go over Ch. 1–5
- Study new material:
  - 5.7 Allpass functions
- Try to make exercise ...

Next lecture, we revisit sampling (Ch. 6).