Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Circuits and Systems

Resit exam EE2S31 SIGNAL PROCESSING
July 31, 2020
Block 1: Stochastic Processes (13:30-15:00)

Open book, strictly timed take-home exam. (Electronic) copies of the book and the course slides allowed. No other tools except a basic pocket calculator permitted.
Upload answers during 14:55–15:10

This block consists of four questions (29 points); more than usual, and this will be taken into account during grading. Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.
Hint: Avoid losing too much time on detailed calculations, write down the general approach first.

Question 1 (8 points)

Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} c & -1 \le x \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the constant c.

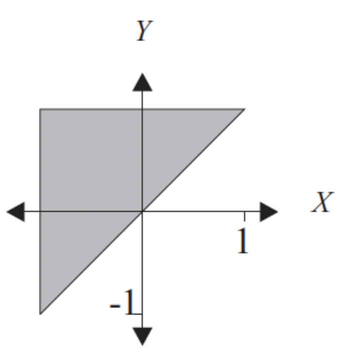
(b) Find $f_Y(y)$ and $f_{X|Y}(x|y)$.

(c) Find $\mathrm{E}[Y]$.

(d) Find $\hat{X}_M(Y)$, the minimum mean square error (MMSE) estimator for X given a single sample of Y.

(e) Find $\hat{Y}_{\mathrm{ML}}(X)$, the maximum likelihood estimator for Y given a single sample of X.

Solution



(a) 1 pnt

$$\begin{aligned}
\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx &= \int_{y=-1}^{1}\int_{x=-1}^{y} c\,dx\,dy \\
&= c\int_{-1}^{1}(y+1)\,dy \\
&= c\left[\frac{1}{2}y^2+y\right]_0^1\,dx \\
&= 2c
\end{aligned}$$

Hence $c=\frac{1}{2}$.

(b) 2 pnt For $-1\le y\le 1$,

$$f_Y(y) = \int_{x=-1}^{y} c\,dx = c(y+1)\,, \qquad -1\le y\le 1$$

The complete PDF is

$$f_Y(y)=\begin{cases}\frac{1}{2}(y+1) & -1\le y\le 1\\ 0 & \text{otherwise.}\end{cases}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases}\dfrac{1}{y+1} & -1\le x\le y\\ 0 & \text{otherwise}\end{cases}$$

(c) 1 pnt

$$\begin{aligned}
\mathrm{E}[Y] &= \int_{-\infty}^{\infty} y\,f_Y(y)\,dy = c\int_{-1}^{1} y(y+1)\,dy \\
&= \frac{1}{2}\left[\frac{1}{3}y^3+\frac{1}{2}y^2\right]_{-1}^{1} = \frac{1}{3}
\end{aligned}$$

(d) 2 pnt The MMSE is always $\hat{X}_M(Y)=\mathrm{E}[X|Y]$. Using $f_{X|Y}(x|y)$,

$$\begin{aligned}
\mathrm{E}[X|Y=y] &= \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx \\
&= \int_{-1}^{y} x\frac{1}{1+y}\,dx \\
&= \frac{1}{1+y}\left[\frac{1}{2}x^2\right]_{x=-1}^{y} \\
&= \frac{1}{1+y}\left(\frac{1}{2}y^2-\frac{1}{2}\right) \\
&= \frac{1}{2}(y-1)
\end{aligned}$$

Then $\hat{X}_M(Y)=\mathrm{E}[X|Y]=\dfrac{1}{2}\,(Y-1)$.

(e) 2 pnt By definition, $\hat{y}_{\mathrm{ML}}(x) = \arg\max_y f_{X|Y}(x|y)$. Then

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{y+1} & -1 \leq x \leq y \\ 0 & \text{otherwise} \end{cases}$$

which should be seen as a function on y. For any x, this function has a support on $x \leq y \leq 1$.

$$\arg\max_{y, y \geq x} \frac{1}{y+1} = x$$

Hence, $\hat{Y}_{\mathrm{ML}}(X) = X$.

Question 2 (7 points)

We flip a coin n times and count how many times we observe "heads"; let $\hat{P}_n$ be the estimated probability of heads (the relative frequency of occurence).

Assume the coin is fair; the true probability of heads is $p = 0.5$.

(a) Find $\mathrm{E}[\hat{P}_n]$ and $\mathrm{Var}[\hat{P}_n]$.

(b) Use the Chebyshev inequality to find an upper bound on the probability that $\hat{P}_{100}$ is within 0.1 of p.

(c) Is $\hat{P}_n$ a consistent estimator of p? (Motivate.)

(d) For large n, we assume that $\hat{P}_n$ can be approximated by a Gaussian distribution. Use this to estimate the probability that $\hat{P}_{100}$ is within 0.1 of p.

Note: See Table 4.1 or 4.2 (page 129/130) for $\Phi(z)$ or $Q(z)$.

Solution

(a) 2 pnt Let A be the event of "heads", then the indicator function X_A has a Bernouilli ($p = 0.5$) random variable.

$$\mathrm{E}[X_A] = p = 0.5, \qquad \mathrm{var}[X_A] = p(1-p) = 0.25\,.$$

Let X_i denote X_A on the ith trial, and let $M_n(X_A)$ be the sample mean (for n samples). Since

$$\hat{P}_n = M_n(X_A) = \frac{1}{n}\sum_{i=1}^{n} X_i$$

is a sum of n independent random variables,

$$\mathrm{var}[\hat{P}_n] = \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{Var}[X_i] = \frac{p(1-p)}{n} = \frac{0.25}{n}$$

(b) 2 pnt Since $\hat{P}_{100}(A) = M_{100}(X_A)$, we can use Theorem 10.5(b) to write (with $c = 0.1$)

$$\mathrm{P}\left[|\hat{P}_{100} - p| < c\right] \geq 1 - \frac{\mathrm{Var}[\hat{P}_n]}{c^2} = 1 - \frac{0.25}{100c^2} = 0.75$$

(c) 1 pnt See Definition 10.3. Here,

$$\mathrm{P}\left[|\hat{P}_n - p| \geq \epsilon\right] \leq \frac{\mathrm{Var}[X_A]}{nc^2}$$

so that

$$\lim_{n\to\infty} \mathrm{P}\left[|\hat{P}_n - p| \geq \epsilon\right] = 0$$

(d) 2 pnt Let $\mu = p = 0.5$ and $\sigma = \sqrt{\frac{0.25}{n}} = \frac{1}{20}$ and $c = 0.1$.

$$\mathrm{P}\left[|\hat{P}_n - p| < c\epsilon\right] = \mathrm{P}\left[-c < \hat{P}_n - p < c\right] = \mathrm{P}\left[-\frac{c}{\sigma} < \frac{\hat{P}_n - \mu}{\sigma} < \frac{c}{\sigma}\right] = \mathrm{P}\left[-\frac{c}{\sigma} < Z < \frac{c}{\sigma}\right] \approx 2\Phi(\frac{c}{\sigma}) - 1$$

With $\frac{c}{\sigma} = 2$, we find (table 4.2) that $\Phi(2) = 0.97725$, so that this probability is 0.95450.

Question 3 (8 points)

Let W be an exponentially distributed random variable, with pdf

$$f_W(w) = \begin{cases} e^{-w} & w \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Consider the random process $X(t) = t - W$.

(a) Draw three realizations of $X(t)$.

(b) Determine the CDF $F_W(w)$ and $F_{X(t)}(x)$, and the pdf of $X(t)$.

(c) Determine $\mathrm{E}[W]$ and compute the expected value function, $\mu_{X(t)}$.

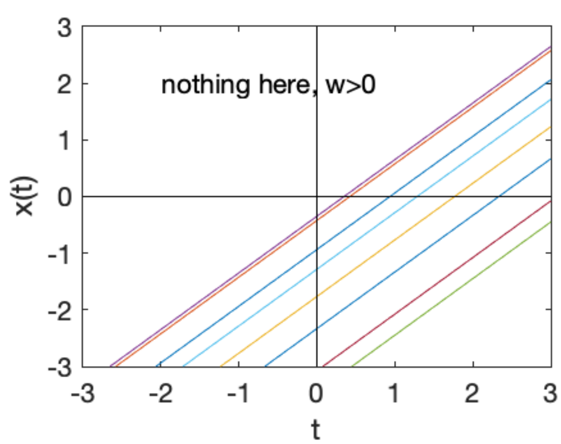
(d) Determine $\mathrm{E}[W^2]$ and compute the autocovariance function, $C_X(t, \tau)$.

(e) Is $X(t)$ a WSS random process? (Motivate)

(f) Is it an i.i.d. process? (Motivate)

Solution

(a) 1 pnt Pick w following an exponential distribution (hence $w > 0$, then plot $X(t) = t - w$ as function of t:



(b) 2 pnt W has an exponential distribution with $\lambda = 1$. (See Thm. 4.8)

$$F_W(w) = \mathrm{P}[W < w] = \begin{cases} 1 - e^{-w} & w \geq 0 \\ 0 & w < 0 \end{cases}$$

$$\begin{aligned} F_{X(t)}(x) &= \mathrm{P}[X < x] = \mathrm{P}[t - W < x] = \mathrm{P}[W > t - x] \\ &= 1 - \mathrm{P}[W < t - x] = \begin{cases} e^{x-t)} & x \leq t \\ 1 & x > t \end{cases} \end{aligned}$$

$$f_{X(t)}(x) = \frac{d\,F_{X(t)}(x)}{d\,x} = \begin{cases} e^{x-t)} & x \leq t\,, \\ 0 & x > t \end{cases}$$

(c) 1.5 pnt For an exponential distribution with $\lambda = 1$, $\mathrm{E}[W] = 1$

$$\mu_X(t) = \mathrm{E}[t - W] = t - \mathrm{E}[W] = t - 1\,.$$

(d) 1.5 pnt $\mathrm{E}[W^2] = 2$.

$$\begin{aligned} C_X(t,\tau) &= \mathrm{E}[X(t)X(t+\tau)] - \mu_X(t)\mu_X(t+\tau) \\ &= \mathrm{E}[(t-W)(t+\tau-W)] - (t-1)(t+\tau-1) \\ &= t(t+\tau) - \mathrm{E}[(2t+\tau)W] + \mathrm{E}[W^2] - t(t+\tau) + 2t + \tau - 1 \\ &= -(2t+\tau)\mathrm{E}[W] + 2 + 2t + \tau - 1 \\ &= 1 \end{aligned}$$

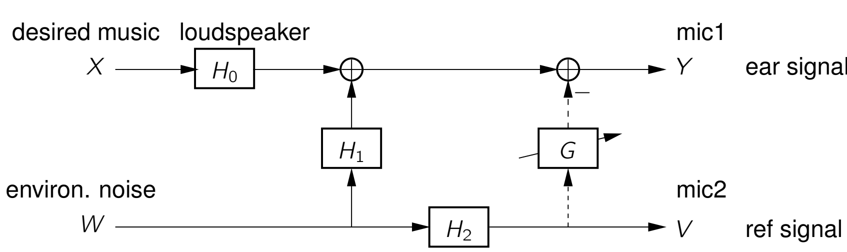
(e) 1 pnt Not WSS because $\mu_X(t)$ is dependent on t.

(f) 1 pnt Not iid because samples are clearly correlated to each other (not independent) and samples do not have the same distribution (it depends on t).

Question 4 (6 points)

In this question, all signals are considered in the frequency domain.

The fundamentals behind a noise canceling headphone are schematically drawn in the figure.



We wish to listen to music $X(f)$ transmitted over a loudspeaker with unknown response $H_0(f)$, but the ear signal $Y(f)$ is disturbed by unknown environment noise $W(f)$, which has been filtered by an unknown channel response $H_1(f)$. We measure the ear signal with microphone 1.

An additional microphone (mic2) also captures the noise signal, but it is filtered by an unknown filter $H_2(f)$.

We wish to design a filter $G(f)$ such that the noise signal on Y is perfectly canceled. **While we design $G(f)$, it is not included in the schematic.**

(a) What is the desired solution for $G(f)$ in terms of $H_1(f)$ and $H_2(f)$?

(b) Show that $H_2^{-1}(f) = |H_2(f)|^{-2} H_2^*(f)$.

$X(f)$ and $W(f)$ are considered to be independent random processes, with power spectral densities $S_X(f)$ and $S_W(f)$, respectively.

(c) Give expressions for $S_Y(f)$, $S_V(f)$ and $S_{YV}(f)$ in terms of $S_X(f)$ and $S_W(f)$.

(d) Which of these (cross) power spectral densities can we observe?

(e) Give an expression for $G(f)$ in terms of observed quantities.

Solution

(a) 1 pnt $G(f) = H_2^{-1}(f) H_1(f)$.

(b) 1 pnt It is the inverse because $H_2^{-1}(f) H_2(f) = |H_2(f)|^{-2} H_2^*(f) H_2(f) = 1$.

(c) 2 pnt

$$\begin{aligned}
S_Y(f) &= |H_0(f)|^2 S_X(f) + |H_1(f)|^2 S_W(f) \\
S_V(f) &= |H_2(f)|^2 S_W(f) \\
S_{YV}(f) &= H_1 H_2^* S_W(f)
\end{aligned}$$

(d) 1 pnt Using the microphone signals, we can observe $Y(f)$ and $V(f)$, and estimate $S_Y(f)$, $S_V(f)$ and $S_{YV}(f)$.

Presumably we also know the input signal $X(f)$ and know $S_X(f)$. But this is not used here. In actuality, $X(f)$ is also disturbed by an unknown filter (not drawn).

(e) 1 pnt

$$G(f) = S_V^{-1}(f) S_{YV}(f)$$

In practice, there are several complications: some signal $X(f)$ may leak into $V(f)$ causing cancellation of the desired signal, $H_2(f)$ may have a zero at some frequency, preventing inversion, and $H_2(s)$ may have zeros in the right hand s-plane, these become unstable poles when we invert the channel.

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Resit exam EE2S31 SIGNAL PROCESSING
July 31, 2020
Block 2: Digital Signal Processing (15:00-16:30)

Open book, strictly timed take-home exam. (Electronic) copies of the book and the course slides allowed. No other tools except a basic pocket calculator permitted. Upload answers during 16:30-16:45.

This block consists of three questions (25 points), more than usual. This will be taken into account during grading. Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. No points will be awarded for results without a derivation. Write your name and student number on each sheet.

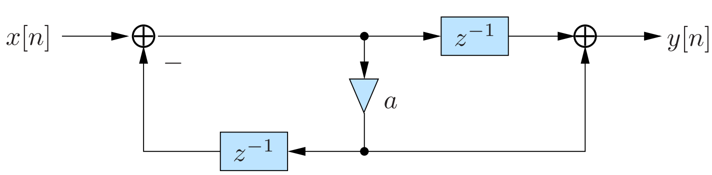
Question 5 (7 points)

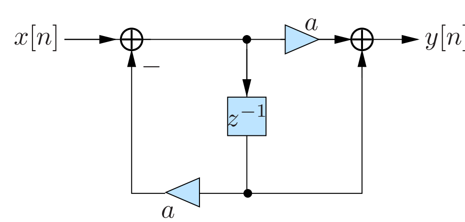
A transfer function $H(z)$ is an allpass function if it satisfies $|H(e^{j\omega})|^2 = 1$. Let $h[n]$ be the corresponding impulse response.

(a) Use a simple argument to show that, for any allpass function, $\sum_n |h[n]|^2 = 1$.

Given two realizations of a first-order allpass function:

$$H(z) = \frac{a + z^{-1}}{1 + az^{-1}}\,, \qquad |a| < 1$$





In the realization in hardware, each multiplier rounds the resulting product to a finite number of bits, such that the roundoff error is in the interval $[-\frac{\Delta}{2}, \frac{\Delta}{2}]$.

This is modeled by introducing a noise source $e_i[n]$ at the output of each multiplier ($i = 1, 2, 3$). Assume that all roundoff errors are iid random variables with uniform distribution.

(b) What is the variance of the equivalent noise source at the output of each multiplier?

(c) Determine the transfer function $H_i(z)$ of each $e_i[n]$ to the output $y[n]$.

(d) For each realization, determine the variance of the quantization noise at the output $y[n]$. Which realization is preferred?

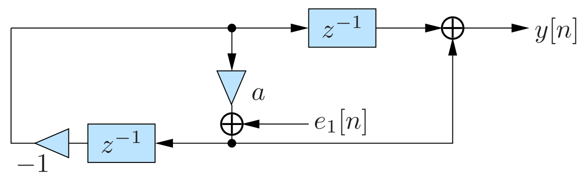
(e) What are two other aspects based on which we would prefer one realization over the other?

Solution

(a) 1 pnt Use Parseval: $\sum_n |h[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 d\omega = 1$.

(b) 1 pnt $\sigma_e^2 = \frac{\Delta^2}{12}$.

(c) 2 pnt For the first realization, the transfer function $H_1(z)$ of $e_1[n] \to y[n]$ is:



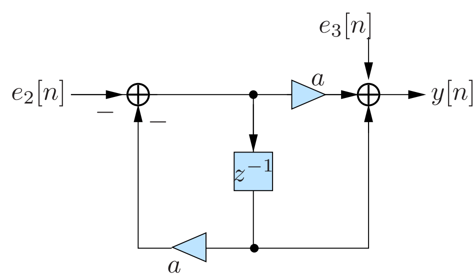
$$Y(z) = E_1(z)\frac{1 - z^{-2}}{1 + az^{-1}} \quad \to \quad H_1(z) = \frac{1 - z^{-2}}{1 + az^{-1}} = 1 - az^{-1} - \frac{1 - a^2}{1 + az^{-1}} \cdot z^{-2}$$

The noise power is given by $\sigma_y^2 = \frac{\Delta^2}{12}\sum(h_1[n])^2$, i.e.,

$$\sigma_y^2 = \frac{\Delta^2}{12}\left(1 + a^2 + (1 - a^2)^2(1 + a^2 + a^4 + \cdots)\right) = 2\frac{\Delta^2}{12}$$

This result also follows more directly by using $H_1(z) = 1 - z^{-1}H(z)$, and the fact that $\sum_n |h[n]|^2 = 1$.

(d) 2 pnt For the direct realization,



$$\begin{aligned} H_2(z) &= -H(z) = -\frac{a + z^{-1}}{1 + az^{-1}} = -\left[a + z^{-1}(1 - a^2)(1 - az^{-1} + a^2 z^{-2} + \cdots)\right] \\ H_3(z) &= 1 \end{aligned}$$

The quantization noise power at the output is

$$\sigma_y^2 = \frac{\Delta^2}{12}\left(1 + a^2 + (1 - a^2)^2(1 + a^2 + a^4 + \cdots\right) = 2\frac{\Delta^2}{12}$$

which can also be obtained more directly using $\sum_n |h_2[n]|^2 = 1$. It follows that both realizations have the same quantization noise power at the output (which is not a priori obvious).

(e) 1 pnt

- The first realization only depends on 1 parameter, and will always be an allpass, even if a is quantized.
- The first realization only uses 1 multiplier and will probably be simpler or faster to implement
- For each new incoming sample, the first realization will require 3 clock beats to produce a new output sample. The direct realization will require 4 clock beats.
- The second realization is minimal.

Question 6 (9 points)

(a) Why would we ever want to use a circular convolution?

(b) Compute the circular convolution (of length 5) of the signal $h[n] = [\boxed{1}, 2, 0, 0, 0]$ with $x[n] = [\boxed{1}, 2, 3, 4, 5]$. (Show the steps in the calculation.)

(c) If $h[n]$ has length L samples, and $x[n]$ has M samples, then how can we use a circular convolution to implement a linear (normal) convolution?

We sample an audio signal $x_a(t)$ at 40 kHz, resulting in $x[n] = x_a(nT)$. We sample during 2.5 seconds. Next, we filter the signal by an FIR filter $h[n]$ that has 1024 coefficients.

(d) What is the maximal permissible frequency in the signal $x_a(t)$?

(e) How many multiplications do we need for a direct implementation of the convolution $y = h * x$? (You may ignore edge effects.)

(f) To reduce the number of operations, we implement the convolution by an FFT using the overlap-add technique.

Describe step-by-step how this can be done. Also draw a block diagram of this process.

(g) How many multiplications are needed now?

[Assume that an FFT of order N has a complexity of $N \log_2(N)$ multiplications.]

(h) As a side effect of this technique, we can simply produce a time-frequency diagram of $x[n]$. What is the resolution in time and in frequency in this diagram?

Solution

(a) 1 pnt If we want to use an FFT to efficiently implement (linear) convolution in frequency domain, then we actually obtain a circular convolution

(b) 1 pnt

$$\begin{array}{lrl} k=0: & h[0]x[n] = & [1 \quad 2 \quad 3 \quad 4 \quad 5] \\ k=1: & h[1]x[(n-1)_5] = & [10 \quad 2 \quad 4 \quad 6 \quad 8] \\ \hline & y = & [11 \quad 4 \quad 7 \quad 10 \quad 13] \end{array}$$

(c) 1 pnt Zero pad both signals to a common length $N = L + M - 1$. In that case, the periodic shift does not give overlap in the output signal.

(d) 1 pnt 20 kHz

(e) 1 pnt In 2.5 seconds, we have 100 kS (kilo-samples). For each sample, we need 1024 multiplications ("flops"). In total: 102 Mflops.

(f) 1 pnt The signal is split in blocks of $M = 1024$ samples. Suppose that $x_i[n]$ is the ith block of 1024 samples, extended with zeros to $N = L + M = 2048$ samples (1 more than strictly necessary). We take the FFT of $x_i[n]$, call it $X_i[k]$. It also has 2048 samples.

The filter $h[n]$ is extended using zero padding to 2048 samples. Let $H[k]$ be its FFT ($N = 2048$ samples).

Next: pointwise multiplication: $Y_i[k] = H[k]X_i[k]$. Determine the IFFT, resulting in $y_i[n]$ consisting of 2048 samples. This covers 2 blocks and overlaps with $y_{i+1}[n]$. The last 1024 samples of $y_i[n]$ overlaps with the first 1024 samples of $y_{i+1}[n]$.

Finally, we use linearity (the response of $\sum x_i[n]$ is $\sum y_i[n]$) and add the overlapping parts of the partial responses.

Block scheme: zie slides DFT ("Overlap-add").

Alternatively, extend the complete x with 1023 zeros, and extend h with 100 000 -1 zeros, and compute $Y[k]$ (length 101 023 samples) using FFT/IFFT. It has a higher complexity and can only be done after the full signal was received (no audio streaming)

(g) 2 pnt FFT of $h[n]$ is done once ($N = 2048$).

FFT of the $x_i[n]$: we have 98 blocks of 1024 samples. We take 98 FFTs each of $N = 2048$ samples.

IFFT of each $y_i[n]$: again 98 FFTs of $N = 2048$ samples.

Total: $1 + 98 + 98 = 197$ times $N \log_2(N)$ is $197 \times 2048 \times 11 = 4.4$ Mflops. Add to this: the frequency domain multiplications, $98 \times N = 200$ kflops. Total: 4.6 Mflops.

(The alternative solution has complexity of the FFTs $3N \log(N)$, with $N = 101024$, gives 5.0 Mflops. Add to this: N multiplications in frequency domain. Total: 5.1 Mflops, not much more!)

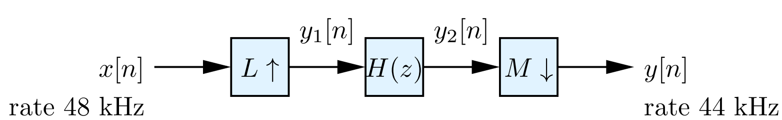
(h) 1 pnt In time, we took FFTs of blocks of 1024 samples. Each block gives 1 time sample. The resolution is $2.5/1024 = 24$ ms.

In frequency, we have 2048 samples spanning 40 kHz, the resolution is $40000/2048 = 19.5$ Hz.

Question 7 (9 points)

An audio signal has frequencies until 20 kHz. We are given $x[n]$ in studio quality: the sample rate is 48 kHz. To transfer this to a CD, we need to reduce the sample rate to 44 kHz.

For this, we are using the following process:



(a) Determine suitable values for the upsampling factor L and downsampling factor M.

(b) What is the role of the filter $H(z)$? Give suitable specifications.

(c) Draw schematically spectra of $x[n]$, $y_1[n]$, $y_2[n]$ and $y[n]$. (Indicate values for both ω and F on the frequency axis.)

(d) Is it allowed to swap the upsampler and downsampler?

(e) Suppose that the required filter $H(z)$ is an FIR filter with 264 coefficients. How many multiplications per second are needed to implement the filter in the shown process?

(f) Can this also be done more efficiently? (If so, indicate how, and how much more efficiently.)

Solution

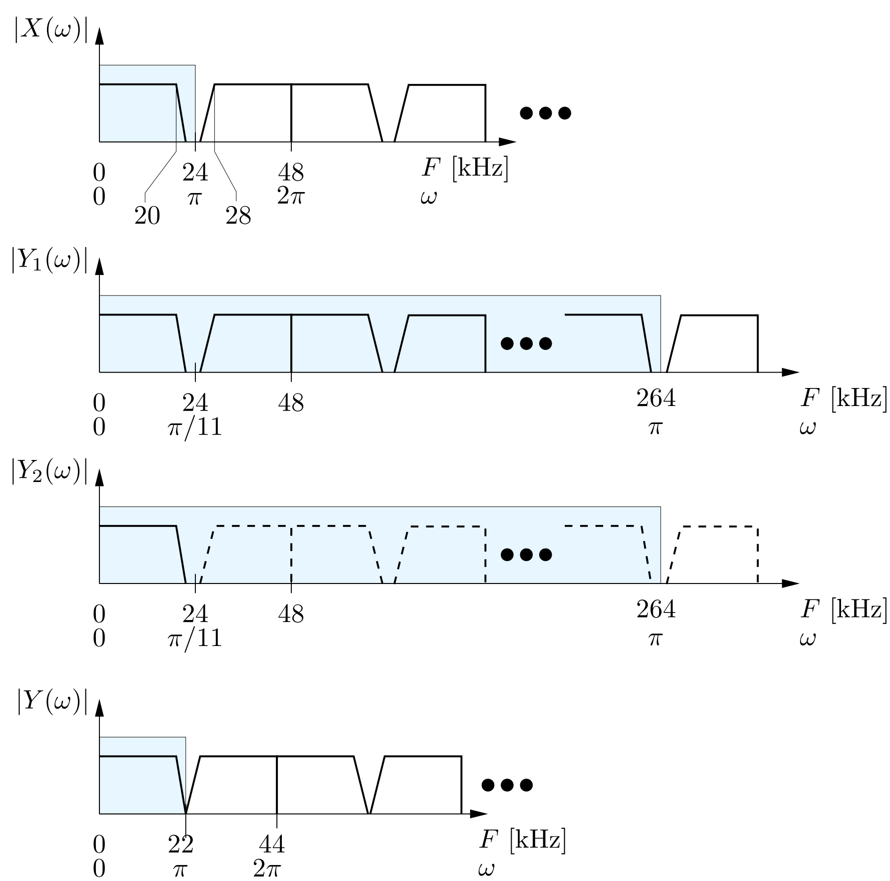
(a) 1 pnt $L = 11,\ M = 12$ so that $48\frac{L}{M} = 44$.

(b) 2 pnt The filter is needed for: (1) removing the additional copies in the spectrum caused by upsampling (remove frequencies above 24 kHz); (2) anti-aliasing before the sample rate reduction / decimation (remove all frequencies above 22 kHz so that resampling at 44 kHz does not cause aliasing).

Combining both functions in one requires $H(z)$ to be a lowpass filter that cuts off at 22 kHz (or $\omega = \frac{\pi}{12}$).

Since the original signal only has frequencies until 20 kHz, we can use a filter with passband until 20 kHz, transition band 20-28 kHz, stopband above 28 kHz. (As this is a digital filter, you should actually convert these values into ω, taking the local sample rate into account: $11 \cdot 48 = 528$ kS/s).

(c) 3 pnt (The fundamental interval is shaded.)



(d) 1 pnt No: if you first downsample, then you throw away information that later you cannot recover; after decimating by a factor 12, the new sample frequency is 4 kHz, and fatal aliasing will occur.

Note: if $H(z)$ was not present, and since M, L are relatively prime, these two blocks can be swapped. But with $H(z)$ in the middle, it is not allowed.

(e 1 pnt) The filter runs at the high sample rate of $48 \cdot 11 = 528$ kHz. The number of flops/s is $528 \cdot 264 = 139$ Mflops/s.

(f) 1 pnt Yes, you don't need to compute the samples that the downsampler will throw away anyway. (This corresponds to placing the downsampler in front of the filter coefficients.) The filter will then run at 44 kHz, requiring $44 \cdot 264 = 11.6$ Mflops/s, a reduction by a factor of 12.

You could also combine the filter with the upsampler, that will give a reduction by a factor of 11.