EE2S31 Signal Processing – Stochastic Processes

Lecture 6: Filtering stochastic processes – Suppl. 1, 2

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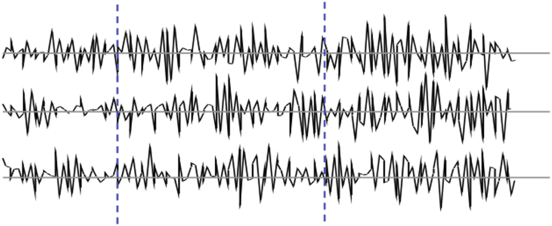
Ch.13.9 Ergodicity

Estimating expected value: ensemble average



How can we estimate $\mathrm{E}[X(n)]$ if we don't know the PDF?

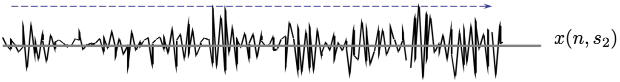
- **Ensemble average:** $\hat{\mu}(n) = \frac{1}{I}\sum_{i=1}^{I} x(n, s_i)$.

  We will need many independent observations!

- If WSS process: $\mathrm{E}[X(n)]$ is the same for all n. Can we use that?

Ergodicity

If the process is **ergodic**, we can also average over time using a single realization (in this case $x(n, s_2)$):



$$\hat{\mu} = \frac{1}{N} \sum_{n=1}^{N} x(n, s_i)$$

Definition: for an *ergodic process*, the time average $\bar{X}$ and the ensemble average $E[X]$ are the same.

Ergodicity

Definition:

For a stationary random process $X(t)$, define the time averages of a sample function $x(t)$ as

$$\begin{aligned} \bar{X}(T) &= \frac{1}{2T}\int_{-T}^{T} x(t)\mathrm{d}t \\ \overline{X^2}(T) &= \frac{1}{2T}\int_{-T}^{T} x^2(t)\mathrm{d}t \end{aligned}$$

These can be measured from a single available observation.

By definition, for an ergodic process

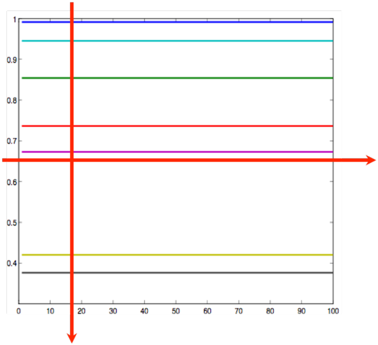
$$\lim_{T\to\infty} \bar{X}(T) = \mu_X$$

- WSS is not sufficient! The autocovariance $C_X(\tau)$ must go to zero quickly enough, so time samples are sufficiently independent.

Ergodicity

Not all WSS processes are ergodic!

Process: $X_n = A$, with random amplitude A, uniform in $[0, 1]$.



time average

ensemble average

$$\begin{aligned} \mu_X &= 0.5 \\ C_X[k] &= E[X_n X_{n+k}] - E[X_n]E[X_{n+k}] \\ &= \text{var}[A] = \frac{1}{12} \end{aligned}$$

Ergodicity

Theorem 13.13 Let $X(t)$ be stationary, with expected value μ_X and autocovariance $C_X(\tau)$.

If $\int_{-\infty}^{\infty} |C_X(\tau)| \mathrm{d}\tau < \infty$, then the sequence $\bar{X}(T), \bar{X}(2T), \cdots$ is an unbiased, consistent sequence of estimators of μ_X.

- It suffices that $C_X(0) < \infty$ (finite variance) and $C_X(\tau) = 0$ for $\tau > \tau_0$.

Proof

- Unbiased:

$$\begin{aligned} \mathrm{E}[\bar{X}(T)] &= \frac{1}{2T} \mathrm{E}\left[\int_{-T}^{T} X(t)\mathrm{d}t\right] = \frac{1}{2T}\int_{-T}^{T} \mathrm{E}[X(t)]\mathrm{d}t \\ &= \frac{1}{2T}\int_{-T}^{T} \mu_X \mathrm{d}t = \mu_X \end{aligned}$$

Ergodicity

Proof (continued)

- Consistent: sufficient to show that $\lim_{T\to\infty} \text{var}[\bar{X}(T)] = 0$:

$$
\begin{aligned}
\text{var}[\bar{X}(T)] &= \text{E}\left[\left(\frac{1}{2T}\int_{-T}^{T}(X(t)-\mu_X)\text{d}t\right)^2\right] \\
&= \frac{1}{(2T)^2}\text{E}\left[\left(\int_{-T}^{T}(X(t)-\mu_X)\text{d}t\right)\left(\int_{-T}^{T}(X(t'-\mu_X)\text{d}t'\right)\right] \\
&= \frac{1}{(2T)^2}\int_{-T}^{T}\int_{-T}^{T}\text{E}[(X(t)-\mu_X)(X(t')-\mu_X)]\text{d}t'\text{d}t \\
&= \frac{1}{(2T)^2}\int_{-T}^{T}\underbrace{\int_{-T}^{T}C_X(t-t')\text{d}t'}_{\text{bounded by some } K}\;\text{d}t
\end{aligned}
$$

Ergodicity

Proof (continued)

Note that

$$\int_{-T}^{T} C_X(t-t')\mathrm{d}t' \leq \int_{-\infty}^{\infty} |C_X(\tau)|\mathrm{d}\tau < \infty$$

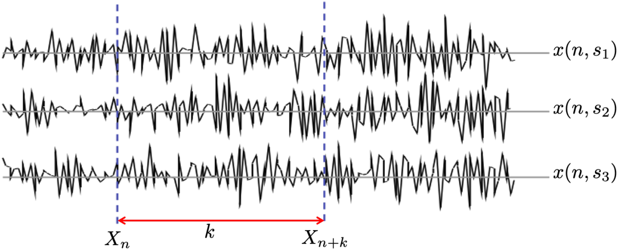
so that there is a constant K such that

$$\mathrm{var}[\bar{X}(T)] \leq \frac{1}{(2T)^2}\int_{-T}^{T} K\mathrm{d}t = \frac{K}{2T}$$

Thus $\lim_{T\to\infty} \mathrm{var}[\bar{X}(T)] \leq \lim_{T\to\infty} \frac{K}{2T} = 0$.

Similar for the Autocorrelation Function (1)

Ensemble average: $$\hat{R}_X[k] = \frac{1}{I}\sum_{i=1}^{I} x(n, s_i)x(n+k, s_i)$$

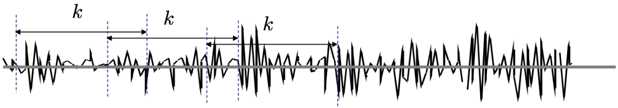


Because the process is WSS, the value of n is not important.

Similar for the Autocorrelation Function (2)

Using time averages, the autocorrelation function can be estimated from a single observation as

$$\bar{R}_X[k] = \frac{1}{N}\sum_{n=1}^{N} x(n, s_i)x(n+k, s_i)$$



Similar for the Autocorrelation Function (3)

The basic estimator form for time averages

$$\bar{R}_X[k] = \frac{1}{N}\sum_{n=1}^{N} x(n, s_i)x(n+k, s_i)$$

uses $2N-1$ data samples to estimate N lags of $R_K[k]$.

- Example for $k = 0,\ 1,\ 2$ and $N = 3$:

$$\begin{aligned} R_X[0] &= \frac{1}{3}\{x(1)^2 + x(2)^2 + x(3)^2\} \\ R_X[1] &= \frac{1}{3}\{x(1)x(2) + x(2)x(3) + x(3)x(4)\} \\ R_X[2] &= \frac{1}{3}\{x(1)x(3) + x(2)x(4) + x(3)x(5)\} \end{aligned}$$

  Also set $R_X[-1] = R_X[1]$, $R_X[-2] = R_X[2]$.

Similar for the Autocorrelation Function (4)

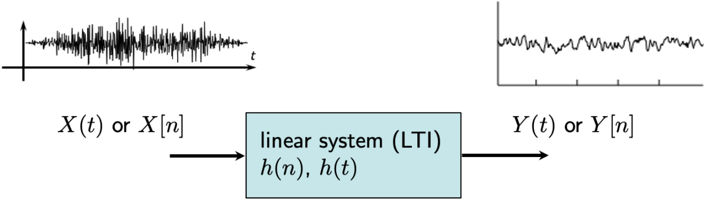
- Modified estimator (using N samples to estimate N correlation lags):

$$\hat{R}_X[k] = \frac{1}{N} \sum_{n=1}^{N-k} x(n, s_i) x(n+k, s_i)$$

$$\begin{aligned} R_X[0] &= \frac{1}{3}\{x(1)^2 + x(2)^2 + x(3)^2\} \\ R_X[1] &= \frac{1}{3}\{x(1)x(2) + x(2)x(3)\} \\ R_X[2] &= \frac{1}{3}\{x(1)x(3)\} \end{aligned}$$

- This estimator is biased: $\mathrm{E}[\hat{R}_X[k]] = \frac{N-k}{N} R_X[k]$
- Unbiased version: $\tilde{R}_X[k] = \dfrac{1}{N-k} \sum_{n=1}^{N-k} x(n, s_i) x(n+k, s_i)$

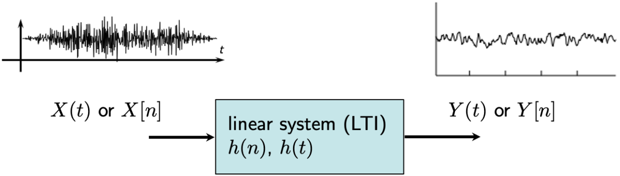
Suppl. 1, 2: Linear filtering of stochastic processes



Signals are often represented as sample functions of WSS processes:

- Use PDF/PMF to describe the amplitude characteristics
- Use autocorrelation to describe the time/spatial varying nature of the signals.

Linear filtering stochastic processes



If the input is a sample function $x(t)$ of a random process $X(t)$ we get

$$y(t) = \int_{-\infty}^{\infty} h(u)x(t-u)\mathrm{d}u \; = \; h(t) * x(t)$$

and therefore we write

$$Y(t) = \int_{-\infty}^{\infty} h(u)X(t-u)\mathrm{d}u \; = \; h(t) * X(t)$$

Expected value of the output

In general:

$$\begin{aligned} \mathrm{E}[Y(t)] &= \mathrm{E}\left[\int_{-\infty}^{\infty} h(u)X(t-u)\mathrm{d}u\right] = \int_{-\infty}^{\infty} h(u)\mathrm{E}\left[X(t-u)\right]\mathrm{d}u \\ &= h(t) * \mathrm{E}[X(t)] \end{aligned}$$

If $X(t)$ is WSS, then $\mathrm{E}[X(t)] = \mu_X$ is constant:

$$\mathrm{E}[Y(t)] = \mu_X \int_{-\infty}^{\infty} h(u)\mathrm{d}u$$

Crosscorrelation (WSS input)

Next, we look at the autocorrelation of $Y(t)$, and crosscorrelation of $X(t)$ with $Y(t)$.

It is convenient to first compute the crosscorrelation:

$$\begin{aligned} R_{XY}(\tau) &= \mathrm{E}[X(t)Y(t+\tau)] \\ &= \mathrm{E}\left[X(t)\int_{-\infty}^{\infty} h(v)X(t+\tau-v)\,\mathrm{d}v\right] \\ &= \int_{-\infty}^{\infty} h(v)\,\mathrm{E}\left[X(t)X(t+\tau-v)\right]\mathrm{d}v \\ &= \int_{-\infty}^{\infty} h(v)R_X(\tau-v)\,\mathrm{d}v \quad = h(\tau) * R_X(\tau) \end{aligned}$$

Autocorrelation (WSS input)

$$R_{XY}(\tau) \;=\; h(\tau) * R_X(\tau)$$

The autocorrelation of the output is then

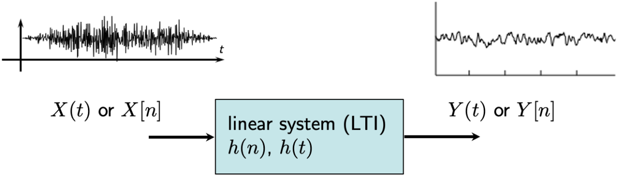
$$\begin{aligned}
R_Y(\tau) &= \mathrm{E}[Y(t)Y(t+\tau)] \\
&= \mathrm{E}\left[\int_{-\infty}^{\infty} h(u)X(t-u)\,\mathrm{d}u \int_{-\infty}^{\infty} h(v)X(t+\tau-v)\,\mathrm{d}v\right] \\
&= \int_{-\infty}^{\infty} h(u)\int_{-\infty}^{\infty} h(v)\,\mathrm{E}\left[X(t-u)X(t+\tau-v)\right]\,\mathrm{d}v\mathrm{d}u \\
&= \int_{-\infty}^{\infty} h(u)\int_{-\infty}^{\infty} h(v)R_X(\tau-v+u)\,\mathrm{d}v\mathrm{d}u \\
&= \int_{-\infty}^{\infty} h(u)R_{XY}(\tau+u)\,\mathrm{d}u \quad = \; h(-\tau) * R_{XY}(\tau) \\
&= h(-\tau) * h(\tau) * R_X(\tau)
\end{aligned}$$

Autocorrelation (WSS input)

Hence, if $X(t)$ is WSS, then $Y(t)$ is also WSS: $\mathrm{E}[Y(t)]$ is independent of time, and $R_Y(t, \tau)$ only depends on the shift τ.

Since also $R_{XY}(t, \tau)$ only depends on τ, we conclude that $X(t)$ and $Y(t)$ are *jointly* WSS.

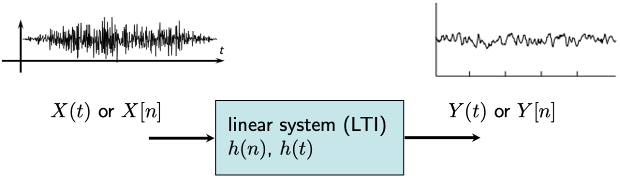
Output distribution



What can we say about the PDF (or PMF) of the output?

- In general this is difficult!
- Exception: a Gaussian stochastic process.

Output distribution



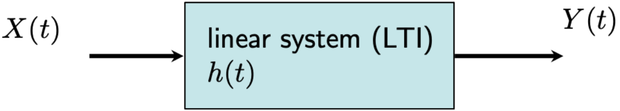
If the input $X(t)$ is a stationary Gaussian stochastic process, and

the filter is LTI with impulse response $h(t)$,

then the output is also stationary Gaussian, with expected value and autocorrelation as specified before.

"Handwaving proof": Remember that a linear transformation of jointly Gaussian RVs gives jointly Gaussian RVs.

Summarizing



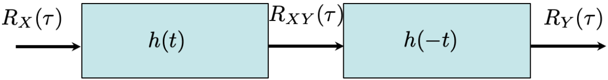
- WSS input gives WSS output

Statistical descriptions of $X(t)$:

- mean μ_X
- Autocorrelation $R_X(\tau)$.

Statistical descriptions of $Y(t)$:

- mean $\mu_Y = \mu_X \int_t h(t)dt$
- $R_Y(\tau) = h(-\tau) * h(\tau) * R_X(\tau)$.



- WSS Gaussian input gives WSS Gaussian output

Example 1

Let $X(t)$ be WSS with $\text{E}[X(t)] = 10$. Apply a linear filter with impulse response

$$h(t) = \begin{cases} e^{t/0.2} & 0 \leq t \leq 0.1 \text{ sec.} \\ 0 & \text{otherwise} \end{cases}$$

Determine $\text{E}[Y(t)]$

Example 1

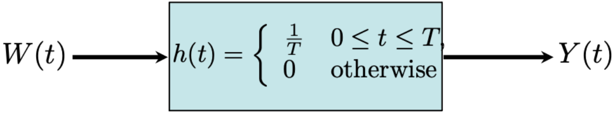
Let $X(t)$ be WSS with $\mathrm{E}[X(t)] = 10$. Apply a linear filter with impulse response

$$h(t) = \begin{cases} e^{t/0.2} & 0 \leq t \leq 0.1 \text{ sec.} \\ 0 & \text{otherwise} \end{cases}$$

Determine $\mathrm{E}[Y(t)]$

$$\mathrm{E}[Y(t)] = \mathrm{E}[X(t)] \int_{-\infty}^{\infty} h(t)\mathrm{d}t = 10 \int_{0}^{0.1} e^{t/0.2}\mathrm{d}t = 2(e^{0.5} - 1)$$

Example 2



$$R_W(\tau) = \eta_0 \delta(\tau)$$

Given $h(t)$ and the white Gaussian noise process $W(t)$ with $R_W(\tau) = \eta_0\, \delta(\tau)$.

Find

- $E[Y(t)]$
- Crosscorrelation $R_{WY}(\tau)$
- Autocorrelation $R_Y(\tau)$

("White" means zero mean, iid)

Example 2

- $\mathrm{E}[W(t)] = 0$ (white Gaussian noise process). So

$$\mathrm{E}[Y(t)] = \mathrm{E}[W(t)] \int_0^T \frac{1}{T} \mathrm{d}t = 0.$$

- Crosscorrelation of input $W(t)$ with output $Y(t)$:

$$\begin{aligned} R_{WY}(\tau) &= \int_{-\infty}^{\infty} h(u) R_W(\tau - u)\,\mathrm{d}u = \frac{\eta_0}{T} \int_0^T \delta(\tau - u)\,\mathrm{d}u \\ &= \begin{cases} \frac{\eta_0}{T} & 0 \le \tau \le T \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

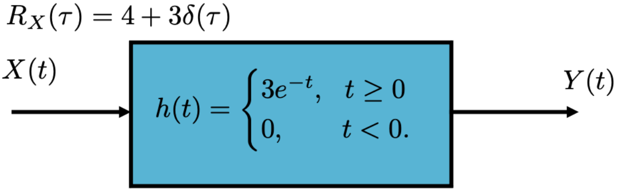
Example 2

- $R_Y(\tau) = \int_{-\infty}^{\infty} h(v) R_{WY}(\tau+v)\,\mathrm{d}v = \int_0^T \frac{1}{T} R_{WY}(\tau+v)\,\mathrm{d}v$
  First write $R_{WY}(\tau+v)$ as function of v:
  $$R_{WY}(\tau+v) = \begin{cases} \frac{\eta_0}{T} & 0 \le \tau+v \le T \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{\eta_0}{T} & -\tau \le v \le T-\tau \\ 0 & \text{otherwise.} \end{cases}$$

- Integration boundaries now depend on τ. Hence, we get two cases:
  $$0 \le \tau \le T \quad : \quad R_Y(\tau) = \frac{1}{T}\int_0^{T-\tau} \frac{\eta_0}{T}\mathrm{d}v = \frac{\eta_0(T-\tau)}{T^2}$$
  $$-T \le \tau \le 0 \quad : \quad R_Y(\tau) = \frac{1}{T}\int_{-\tau}^{T} \frac{\eta_0}{T}\mathrm{d}v = \frac{\eta_0(T+\tau)}{T^2}$$

- Altogether: $R_Y(\tau) = \begin{cases} \frac{\eta_0(T-|\tau|)}{T^2} & |\tau| \le T, \\ 0 & \text{otherwise.} \end{cases}$

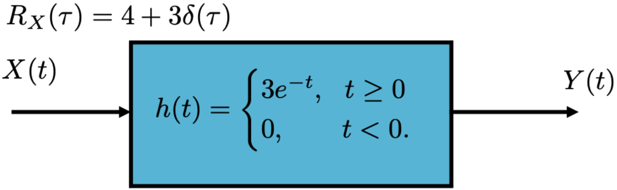
Example 3



$$
\begin{aligned}
R_Y(\tau) &= h(\tau) * h(-\tau) * R_X(\tau) \\
&= g(\tau) * R_X(\tau)
\end{aligned}
$$

$$
\begin{aligned}
g(\tau) = h(\tau) * h(-\tau) &= \int_{-\infty}^{\infty} 3e^{-t} u(t)\, 3e^{-t+\tau} u(-\tau + t)\, \mathrm{d}t \\
&= \begin{cases} 9e^{\tau} \int_{\tau}^{\infty} e^{-2t} \mathrm{d}t = \frac{9}{2} e^{-\tau} & \text{if } \tau \geq 0 \\ 9e^{\tau} \int_{0}^{\infty} e^{-2t} \mathrm{d}t = \frac{9}{2} e^{\tau} & \text{if } \tau < 0 \end{cases}
\end{aligned}
$$

Example 3



$$
\begin{aligned}
R_Y(\tau) = g(\tau) * R_X(\tau) &= \left(\frac{9}{2} e^{-\tau} u(\tau) + \frac{9}{2} e^{\tau} u(-\tau) \right) * (4 + 3\delta(\tau)) \\
&= \int_{-\infty}^{+\infty} \frac{9}{2} \left(e^{-t} u(t) + e^{t} u(-t) \right) (4 + 3\delta(\tau - t)) \mathrm{d}t \\
&= \frac{36}{2} \int_0^{\infty} e^{-t} \mathrm{d}t + \frac{36}{2} \int_{-\infty}^{0} e^{t} \mathrm{d}t + \frac{27}{2} e^{-\tau} u(\tau) + \frac{27}{2} e^{\tau} u(-\tau) \\
&= 36 + \frac{27}{2} e^{-|\tau|}
\end{aligned}
$$

Sampling and filtering of random processes

Let $X(t)$ be a continuous WSS process with $\mathrm{E}[X(t)] = \mu_X$ and $R_X(\tau)$.

Sample with period T_s: $X_n = X(nT_s)$. Then

X_n is also WSS with $\mathrm{E}[X_n] = \mu_X$ and $R_X[k] = R_X(kT_s)$, because

$$\mathrm{E}[X_n] = \mathrm{E}[X(nT_s)] = \mu_X$$

$$R_X[k] = \mathrm{E}[X_n X_{n+k}] = \mathrm{E}[X(nT_s)X([n+k]T_s)] = R_X(kT_s).$$

Filtering of discrete-time random sequences:

$Y_n = h_n * X_n = \sum_j h_j X_{n-j}$

- $\mathrm{E}[Y_n] = \mathrm{E}[X_n] \sum_j h_j$
- $R_{XY}[k] = \mathrm{E}[X_n Y_{n+k}] = \sum_j h_j R_X[k-j] \; = \; h_k * R_X[k]$
- $R_Y[k] = \mathrm{E}[Y_n Y_{n+k}] = \sum_i h_i \sum_j h_j R_X[k+i-j] \; = \; h_{-k} * R_{XY}[k]$

Example

Let Y_n be a sampled version of stochastic process $Y(t)$. $Y(t)$ has autocorrelation function

$$R_Y(\tau) = \begin{cases} 10^{-9}(10^{-3} - |\tau|) & |\tau| \leq 10^{-3}, \\ 0 & \text{otherwise.} \end{cases}$$

What is the autocorrelation function of the sampled process Y_n if $F_s = 10^4$ samples/sec?

Example

Let Y_n be a sampled version of stochastic process $Y(t)$. $Y(t)$ has autocorrelation function

$$R_Y(\tau) = \begin{cases} 10^{-9}(10^{-3} - |\tau|) & |\tau| \leq 10^{-3}, \\ 0 & \text{otherwise.} \end{cases}$$

What is the autocorrelation function of the sampled process Y_n if $F_s = 10^4$ samples/sec?

$$\begin{aligned} R_Y[k] &= R_Y\left(k\frac{1}{F_s}\right) = \begin{cases} 10^{-9}(10^{-3} - |k\frac{1}{F_s}|) & |k\frac{1}{F_s}| \leq 10^{-3} \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 10^{-9}(10^{-3} - |k\,10^{-4}|) & |k\,10^{-4}| \leq 10^{-3}, \\ 0 & \text{otherwise.} \end{cases} \\ &= \begin{cases} 10^{-6}(1 - |0.1k|) & |k| \leq 10, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Problem 2.7 (modified notation)

Consider $X_n = aX_{n-1} + V_n$, where V_n: iid, $\mathrm{E}[V_n] = 0$, $R_V[k] = \sigma^2\delta[k]$.

Find $R_X[k]$.

Problem 2.7 (modified notation)

Consider $X_n = aX_{n-1} + V_n$, where V_n: iid, $\mathrm{E}[V_n] = 0$, $R_V[k] = \sigma^2\delta[k]$.

Find $R_X[k]$.

$$\begin{aligned}
R_{VX}[k] &= \mathrm{E}[V_{n-k}X_n] = \mathrm{E}[V_{n-k}(aX_{n-1} + V_n)] \\
&= aR_{VX}[k-1] + \sigma^2\delta[k] \\
\Rightarrow \; R_{VX}[k] &= \begin{cases} \sigma^2 a^k & k \geq 0, \\ 0 & k < 0. \end{cases}
\end{aligned}$$

$$R_{XV}[k] = R_{VX}[-k]$$

$$\begin{aligned}
R_X[k] &= \mathrm{E}[X_{n-k}X_n] = \mathrm{E}[X_{n-k}(aX_{n-1} + V_n)] \\
&= aR_X[k-1] + R_{XV}[k]
\end{aligned}$$

Problem 2.7 (cont'd)

We saw until now:

$$R_x[k] = aR_X[k-1] + R_{XV}[k]\,, \qquad R_{XV}[k] = \begin{cases} \sigma^2 a^{-k} & k \le 0, \\ 0 & k > 0. \end{cases}$$

$$k > 0: \qquad R_X[k] = aR_X[k-1] \;= \cdots =\; a^k R_X[0]$$

$$k = 0: \qquad R_X[0] = aR_X[-1] + \sigma^2 = aR_X[1] + \sigma^2 = a^2 R_X[0] + \sigma^2$$

$$R_X[0] = \frac{\sigma^2}{1-a^2} \quad =: \quad \sigma_X^2$$

It follows, for $k \ge 0$: $R_X[k] = a^k \sigma_X^2$

Also, for $k < 0$, $R_X[k] = R_X[-k] = a^{-k}\sigma_X^2$

$$\Rightarrow \; R_X[k] = a^{|k|}\frac{\sigma^2}{1-a^2}$$

To do for this lecture:

- Make some selected exercises of the Supplement:
  1.1, 1.3, 2.1, 2.3, 2.5, 2.7
  (Unfortunately, the supplement has far fewer exercises)

Next lecture, we'll do Supplement Sections 5 and 6.