EE2S31 Signal Processing – Stochastic Processes

Lecture 5: Stochastic processes – Ch. 13

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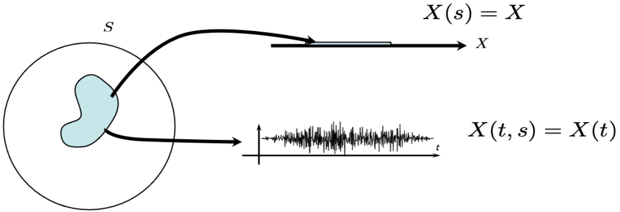
Today: Ch. 13 Stochastic Processes

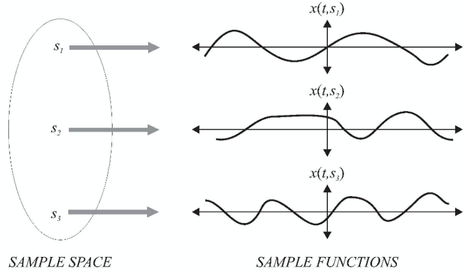
- **Stochastic:** random
- **Process:** sequence of variables where the ordering is of importance.



- **Random variable:** Mapping from an outcome s in the sample space to a real number $x(s)$.
- **Stochastic process:** Mapping from an outcome s in the sample space to a function $x(t, s)$, which depends on an ordering variable like time or space: a *random signal*.

Stochastic Processes

- The stochastic process is denoted by $X(t)$.
- Sample function $x(t, s_1)$ is one particular realization (outcome s_1) of this process.



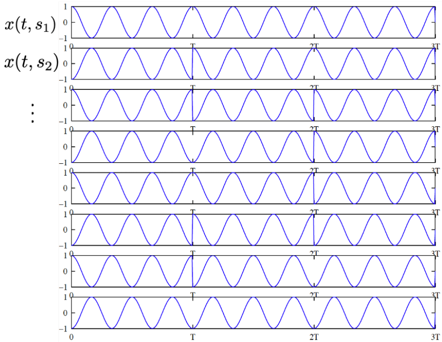
The **ensemble** of a stochastic process is the set of all possible time functions that can result from an experiment.

Problem 13.1.3

Consider the transmission of 3 bits in a BPSK system (binary phase shift keying):

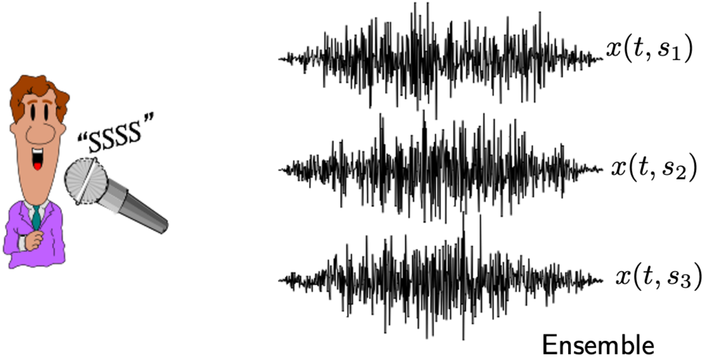
$$S \in \{000, 001, 010, 011, 100, 101, 110, 111\}$$

The ensemble consists of 8 possible sample functions:



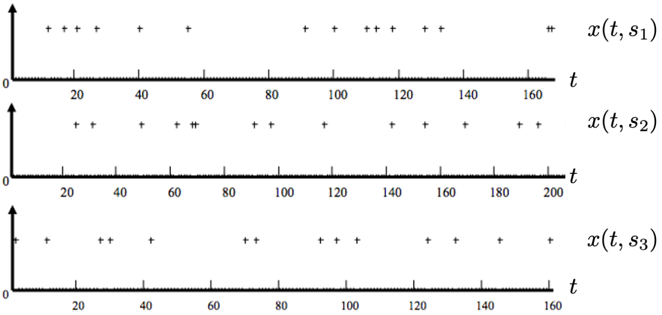
Example: speech

- Experiment 's': pronounce 'ssss'



- Sample function: one realization of the waveform $x(t, s)$

Example: arrival times of data packets



Example: binary bit pattern

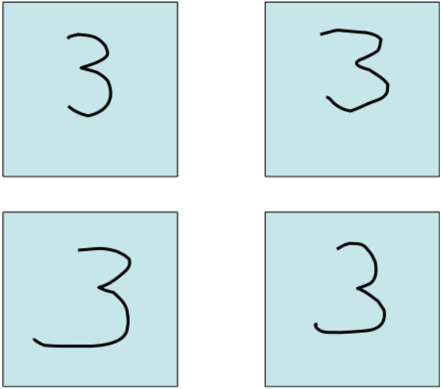
$$X_3 = 000011111100000111101010001110000000000111010$$

$$X_{17} = 011110011100110101111010100011100010010100001$$

$$X_{55} = 000000001111111111110100111111111111100111110$$

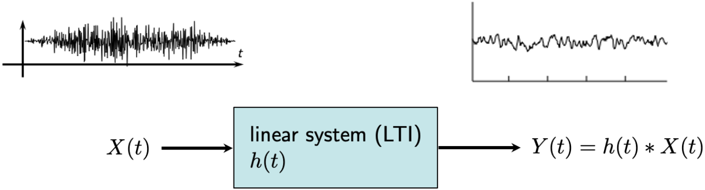
- A stochastic description can be used for image compression.

Example: handwritten digits



- A stochastic description (features!) is used for pattern recognition.

Example: linear filtering of a stochastic process



How can we describe $Y(t)$ when $X(t)$ is a stochastic process?

Statistical descriptions of $X(t)$:

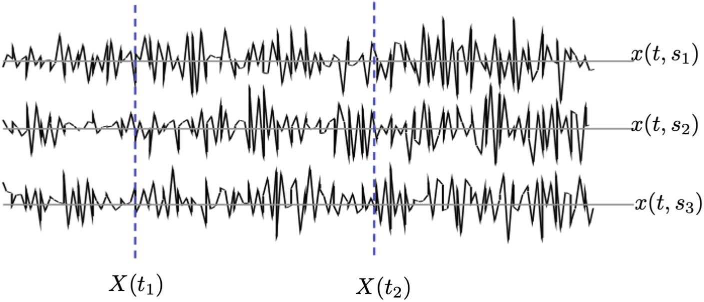
- mean μ_X
- Autocorrelation function $R_X(\tau)$.

Statistical descriptions of $Y(t)$:

- mean $\mu_Y = \mu_X \int_t h(t) \mathrm{d}t$
- $R_Y(\tau) = h(\tau) * h(-\tau) * R_X(\tau)$.

This will be the topic of the Supplement (next weeks)

Description of a random process



- How to describe a stochastic process at one time instance: e.g., $X(t_1)$?
- How to describe a stochastic process at multiple time instances: e.g., $[X(t_1), X(t_2)]^T$?

Description of a random process

Similar as for RVs, we can use the PDF/PMF to describe stochastic processes:

- At any (fixed) time t_k the stochastic process can be regarded as a random variable:

  $$X(t_k) \sim f_{X_{t_k}}(x_{t_k})$$

  This PDF may be different for each t_k!

- The joint behavior for multiple time instances t, i.e., $t_1, \cdots, t_k$ is given by the joint PDF:

$$[X(t_1), \cdots, X(t_k), \cdots]^T \sim f_{X_{t_1}, X_{t_2}, \cdots, X_{t_k}, \cdots}(x_{t_1}, x_{t_2}, \cdots, x_{t_k}, \cdots)$$

Example: rectified sinusoid with random amplitude

Let $X(t) = R|\cos(\omega t)|$ with

$$f_R(r) = \begin{cases} \frac{1}{10} e^{-r/10} & r \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Calculate $f_{X(t)}(x)$!

Approach

- First calculate the CDF $F_{X(t)}(x)$
- Then calculate the PDF $f_{X(t)}(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_{X(t)}(x)$

Example: rectified sinusoid with random amplitude

$$
\begin{aligned}
F_{X(t)}(x) &= \mathrm{P}[X(t) \leq x] \\
&= \mathrm{P}[R\,|\cos(\omega t)| \leq x] \\
&= \mathrm{P}\left[R \leq x/|\cos(\omega t)|\right] \qquad \text{if } \cos(\omega t) \neq 0 \\
&= \int_0^{x/|\cos(\omega t)|} f_R(r)\,\mathrm{d}r \\
&= 1 - e^{-\frac{x}{10|\cos(\omega t)|}} \qquad \text{if } x \geq 0
\end{aligned}
$$

- If $\cos(\omega t) \neq 0$:

$$
F_{X(t)}(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\frac{x}{10|\cos(\omega t)|}} & x \geq 0. \end{cases}
$$

Example: rectified sinusoid with random amplitude

- If $\cos(\omega t) \neq 0$:

$$f_{X(t)}(x) = \frac{\mathrm{d}F_{X(t)}(x)}{\mathrm{d}x} = \begin{cases} 0 & x < 0 \\ \frac{1}{10|\cos(\omega t)|} e^{-\frac{x}{10|\cos(\omega t)|}} & x \geq 0. \end{cases}$$

- If $\cos(\omega t) = 0$, then $X(t) = 0$ (constant) and $f_{X(t)}(x) = \delta(x)$.

Problem 13.2.1 (similar: 13.2.4)

Let W be an exponential random variable with PDF

$$f_W(w) = \begin{cases} e^{-w} & w \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the CDF $F_{X(t)}(x)$ of the time-delayed ramp process $X(t) = t - W$.

Problem 13.2.1 (similar: 13.2.4)

Let W be an exponential random variable with PDF

$$f_W(w) = \begin{cases} e^{-w} & w \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the CDF $F_{X(t)}(x)$ of the time-delayed ramp process $X(t) = t - W$.

$$\mathrm{P}[X(t) \leq x] \; = \; \mathrm{P}[t - W \leq x] \; = \; \mathrm{P}[W \geq t - x]\,.$$

Since $W \geq 0$, if $x \geq t$ then $\mathrm{P}[W \geq t - x] = 1$. When $x < t$,

$$\mathrm{P}[W \geq t - x] \; = \; \int_{t-x}^{\infty} f_W(w)\mathrm{d}w \; = \; e^{-(t-x)}\,.$$

Problem 13.2.1 (cont'd)

Combining the facts, we have

$$F_{X(t)}(x) = \mathrm{P}[W \geq t - x] = \begin{cases} e^{-(t-x)} & x < t \\ 1 & x \geq t \end{cases}$$

We note that the CDF contains no discontinuities. Taking the derivative of the CDF with respect to x, we obtain the PDF

$$f_{X(t)}(x) = \begin{cases} e^{-(t-x)} & x < t \\ 0 & x \geq t \end{cases}$$

- Treat t as a constant
- Calculate CDF $F_{X(t)}(x)$
- Then calculate PDF $f_{X(t)}(x)$

Description of a random process

Notice that

$$[X(t_1), \cdots, X(t_k), \cdots]^T \sim f_{X_{t_1}, X_{t_2}, \cdots, X_{t_k}, \cdots}(x_{t_1}, x_{t_2}, \cdots, x_{t_k}, \cdots)$$

resembles a vector random variable,

- but can be of infinite dimensionality,
- and ordering (in time) of the $X(t_k)$ is essential.

Generally, the joint PDF is very difficult to acquire. Exceptions:

- Independent identically distributed (iid) random sequence/process
- Gaussian stochastic process
- Poisson process (skipped), Brownian motion process (skipped)

Independent identically distributed (iid) random sequences

For an iid random sequence

- all $X(t_k)$ are mutually independent random variables for all t_k,
- all $X(t_k)$ have the same PDF for all t_k.

Let X_n denote a (time discrete) iid random sequence with sample vector $\boldsymbol{X} = [X_{n_1}, \cdots, X_{n_k}]^T$.

- For a discrete valued X_n, the joint PMF is

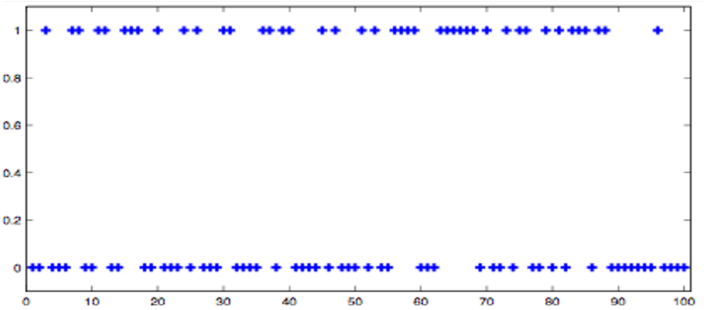
$$P_{\boldsymbol{X}}(\boldsymbol{x}) = P_{X_1}(x_1) \cdots P_{X_k}(x_k) = \prod_{i=1}^{k} P_X(x_i)$$

- For a continuous valued X_n, the joint PDF is

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = f_{X_1}(x_1) \cdots f_{X_k}(x_k) = \prod_{i=1}^{k} f_X(x_i)$$

Example: Bernoulli process (time discrete)

One realization of a Bernoulli process (e.g., a bit sequence)



PMF for one time instance k:

$$P_{X_k}(x_k) = \begin{cases} p & x_k = 1 \\ 1-p & x_k = 0 \\ 0 & \text{otherwise} \end{cases} \Leftrightarrow P_{X_k}(x_k) = \begin{cases} p^{x_k}(1-p)^{1-x_k} & x_k = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

Example: Bernoulli process (cont'd)

$$P_{X_k}(x_k) = \begin{cases} p^{x_k}(1-p)^{1-x_k} & x_k = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

For two time instances t_1 and t_2 we obtain (iid process!)

$$\begin{aligned} P_{X_1,X_2}(x_1, x_2) &= P_{X_1}(x_1)P_{X_2}(x_2) = p^{x_1}(1-p)^{1-x_1}p^{x_2}(1-p)^{1-x_2} \\ &= p^{x_1+x_2}(1-p)^{2-x_1-x_2} \end{aligned}$$

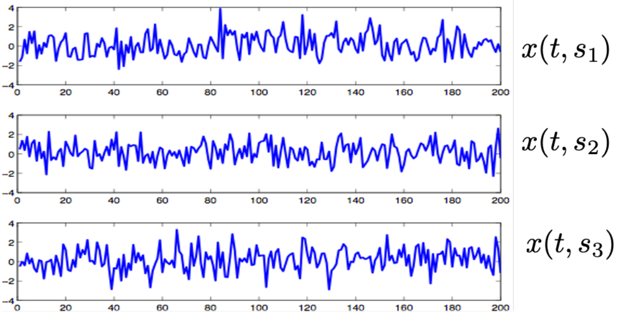
For k time instances we obtain

$$P_{\boldsymbol{X}}(\boldsymbol{x}) = \prod_{i=1}^{k} p^{x_i}(1-p)^{1-x_i} = p^{x_1+\cdots+x_k}(1-p)^{k-(x_1+\cdots+x_k)}$$

Gaussian Process

Gaussian processes occur quite often in nature (remember the central limit theorem!)

The process $X(t)$ is a Gaussian stochastic process (sequence) if and only if $\boldsymbol{X} = [X(t_1) \cdots X(t_k)]^T$ is a Gaussian random vector for any integer $k > 0$ and any set of time instances $t_1, t_2, \cdots, t_k$.



Gaussian Process

Remember that for Gaussian random vectors $\boldsymbol{X} = [X_1, X_2, \cdots, X_N]^T$ we have:

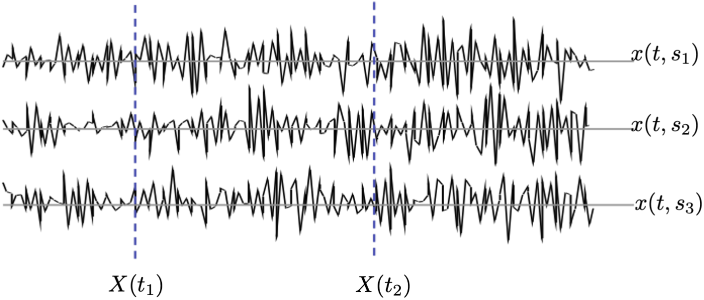
$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{\exp\left[-\frac{1}{2}\left(\boldsymbol{x} - \mathrm{E}[\boldsymbol{X}]\right)^T \boldsymbol{C}_{\boldsymbol{X}}^{-1}\left(\boldsymbol{x} - \mathrm{E}[\boldsymbol{X}]\right)\right]}{(2\pi)^{N/2}\det(\boldsymbol{C}_{\boldsymbol{X}})^{1/2}}$$

For a Gaussian stochastic process $X(t)$, the distribution of $\boldsymbol{X} = [X_{t_1}, X_{t_2}, ..., X_{t_k}]^T$ is thus also given by $f_{\boldsymbol{X}}(\boldsymbol{x})$.

For each collection of sample times $t_1, \cdots, t_k$, we need to specify

- The mean: $\boldsymbol{\mu}_{\boldsymbol{X}} = \mathrm{E}[\boldsymbol{X}]$; specify $\mathrm{E}[X(t_i)]$ for all i;
- The (auto)covariance: $\mathrm{cov}[\boldsymbol{X}, \boldsymbol{X}] = \boldsymbol{C}_{\boldsymbol{X}}$;
  specify $\mathrm{cov}[X(t_i), X(t_j)]$ for all i, j.

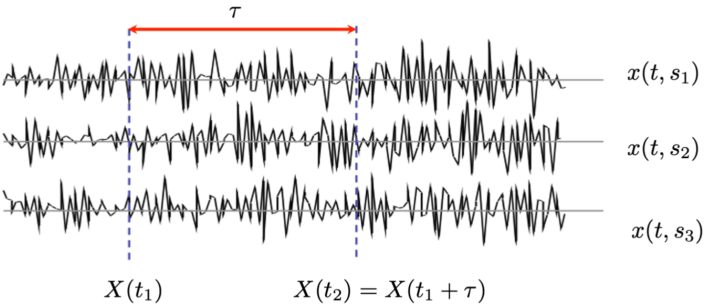
Expected value of a stochastic process



Expected value of $X(t_k)$ at time t_k:

$$\mathrm{E}[X(t)] = \int_{-\infty}^{\infty} x\, f_{X(t)}(x)\, \mathrm{d}x$$

Autocovariance of a stochastic process



- $\mathrm{cov}[X(t_1), X(t_2)]$ indicates how much the process is likely to change from t_1 to t_2.
- Large covariance: sample function unlikely to change.
- Zero covariance: sample function expected to change rapidly.

Autocovariance and autocorrelation

The covariance of a stochastic process at two different time instances is called "autocovariance":

$$
\begin{aligned}
C_X(t,\tau) &= \text{cov}[X(t), X(t+\tau)] \\
&= \mathsf{E}\left[(X(t) - \mathsf{E}[X(t)])\,(X(t+\tau) - \mathsf{E}[X(t+\tau)])\right] \\
&= \underbrace{\mathsf{E}[X(t)X(t+\tau)]}_{\text{Similar to crosscorrelation } \mathsf{E}[XY]} - \mathsf{E}[X(t)]\mathsf{E}[X(t+\tau)]
\end{aligned}
$$

The autocorrelation is similarly defined as

$$
R_X(t,\tau) = \mathsf{E}[X(t)X(t+\tau)]
$$

$$
\begin{aligned}
C_X(t,\tau) &= R_X(t,\tau) - \mathsf{E}[X(t)]\,\mathsf{E}[X(t+\tau)] \\
C_X(t,0) &= \mathsf{E}[X(t)^2] - \mathsf{E}[X(t)]^2 = \text{var}[X(t)]
\end{aligned}
$$

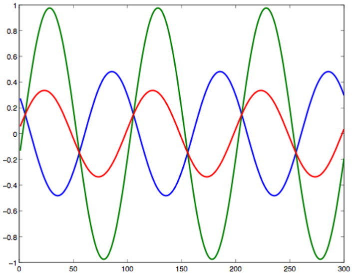
The Autocorrelation function

Notice that exact formulation of the autocorrelation depends on whether time and amplitudes are discrete or continuous:

$$
\begin{aligned}
R_X(t,\tau) &= \mathrm{E}[X(t)X(t+\tau)] = \iint x\,y\,f_{X(t),X(t+\tau)}(x,y)\,\mathrm{d}x\mathrm{d}y \\
R_X(t,\tau) &= \mathrm{E}[X(t)X(t+\tau)] = \sum_x \sum_y xy \mathrm{P}[X(t)=x, X(t+\tau)=y]
\end{aligned}
$$

$$
\begin{aligned}
R_X[n,k] &= \mathrm{E}[X_n X_{n+k}] = \iint x\,y\,f_{X_n,X_{n+k}}(x,y)\,\mathrm{d}x\mathrm{d}y \\
R_X[n,k] &= \mathrm{E}[X_n X_{n+k}] = \sum_x \sum_y xy \mathrm{P}[X_n = x, X_{n+k} = y]
\end{aligned}
$$

Example: sinusoidal process



Random process: $X(t) = A \sin(\omega t + \Phi)$

Amplitude A and phase Φ are independent random variables, where

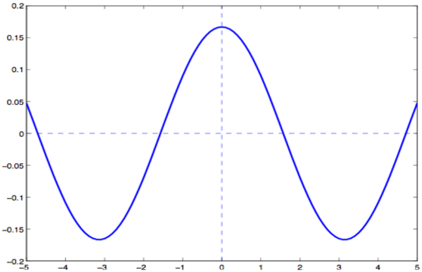
- A is uniformly distributed on $[-1, +1]$
- Φ is uniformly distributed on $[0, 2\pi]$

Example: sinusoidal process

$$
\begin{aligned}
R_X(t,\tau) &= \mathrm{E}[X(t)X(t+\tau)] \\
&= \mathrm{E}[A^2 \sin(\omega t + \Phi)\sin(\omega(t+\tau)+\Phi)] \\
&= \mathrm{E}[A^2]\ \mathrm{E}[\sin(\omega t + \Phi)\sin(\omega(t+\tau)+\Phi)] \\
&= \int_{-1}^{1} \frac{a^2}{2} \mathrm{d}a\ \mathrm{E}[\sin(\omega t + \Phi)\sin(\omega(t+\tau)+\Phi)] \\
&= \frac{1}{3}\mathrm{E}[\sin(\omega t + \Phi)\sin(\omega(t+\tau)+\Phi)] \\
&= \frac{1}{6}\mathrm{E}[\cos(\omega\tau) - \cos(2\omega t + \omega\tau + 2\Phi)] \\
&= \frac{1}{6}\cos(\omega\tau) - \frac{1}{6}\underbrace{\mathrm{E}[\cos(2\omega t + \omega\tau + 2\Phi)]}_{=0} \\
&= \frac{1}{6}\cos(\omega\tau)
\end{aligned}
$$

Example: sinusoidal process

$R_X(t, \tau) = \frac{1}{6}\cos(\omega\tau)$



High correlation for $\omega\tau = 0, \pm 2\pi, \cdots$

Zero correlation for $\omega\tau = \pm\frac{1}{2}\pi, \cdots$

Very negative correlation for $\omega\tau = \pm\pi, \cdots$

Uncorrelated and orthogonal processes

- If all pairs $X(t), X(t+\tau)$ are *uncorrelated*, i.e.,

$$C_X(t,\tau) = \begin{cases} \mathrm{var}[X(t)] & \forall\, t \text{ and } \tau = 0 \\ 0 & \forall\, t \text{ and } \tau \neq 0 \end{cases}$$

  then $X(t)$ is called an uncorrelated process.

- If all pairs $X(t), X(t+\tau)$ are *orthogonal*, i.e.,

$$R_X(t,\tau) = \begin{cases} \mathrm{E}[X^2(t)] & \forall\, t \text{ and } \tau = 0 \\ 0 & \forall\, t \text{ and } \tau \neq 0 \end{cases}$$

  then $X(t)$ is called an orthogonal process

Problem 13.7.2

For the time-delayed ramp process $X(t) = t - W$ from Problem 13.2.1, find for any $t \geq 0$:

(a) The expected value $\mu_X(t)$,

(b) The autocovariance function $C_X(t, \tau)$.
Hint: $\text{E}[W] = 1$ and $\text{E}[W^2] = 2$.

Problem 13.7.2

For the time-delayed ramp process $X(t) = t - W$ from Problem 13.2.1, find for any $t \geq 0$:

(a) The expected value $\mu_X(t)$,

(b) The autocovariance function $C_X(t, \tau)$.
Hint: $E[W] = 1$ and $E[W^2] = 2$.

We know but don't need:

$$f_{X(t)}(x) = \begin{cases} e^{x-t} & x < t \\ 0 & \text{o.w.} \end{cases}$$

(a) The mean is
$\mu_X(t) = E[t - W] = t - E[W] = t - 1.$

(b) The autocovariance is

$$\begin{aligned} C_X(t, \tau) &= E[X(t)X(t+\tau)] - \mu_X(t)\mu_X(t+\tau) \\ &= E[(t - W)(t + \tau - W)] - (t-1)(t+\tau-1) \\ &= t(t+\tau) - E[(2t+\tau)W] + E[W^2] - (t-1)(t+\tau-1) \\ &= -(2t+\tau)E[W] + 2 + 2t + \tau - 1 \ = \ 1 \end{aligned}$$

Stationary Process

A stochastic process is stationary if and only if every joint-PDF is shift invariant:

$$f_{X(t_1),X(t_2),\cdots,X(t_k)}(x_1, x_2, \cdots, x_k)$$
$$= f_{X(t_1+\Delta t),X(t_2+\Delta t),\cdots,X(t_k+\Delta t)}(x_1, x_2, \cdots, x_k)$$

- **Consequence I** The marginal PDF's are independent of t:

$$f_{X(t)}(x) = f_{X(t+\Delta t)}(x) = f_X(x)$$

The marginal PDF's are identical for all t_k!
$\Rightarrow$ Expected value and variance are time independent.

Problem 13.8.2

$\boldsymbol{X} = [X_1 \;\; X_2]^T$ has expected value $\mathrm{E}[\boldsymbol{X}] = 0$ and covariance matrix

$$\boldsymbol{C_X} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Does there exist a stationary process $X(t)$ and time instances t_1 and t_2 such that $\boldsymbol{X}$ is actually a pair of observations $[X(t_1) \;\; X(t_2)]^T$ of the process $X(t)$?

Problem 13.8.2

$\boldsymbol{X} = [X_1 \;\; X_2]^T$ has expected value $\mathrm{E}[\boldsymbol{X}] = 0$ and covariance matrix

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Does there exist a stationary process $X(t)$ and time instances t_1 and t_2 such that $\boldsymbol{X}$ is actually a pair of observations $[X(t_1) \;\; X(t_2)]^T$ of the process $X(t)$?

The short answer is No. For the given process $X(t)$,

$$\mathrm{var}[X(t_1)] = C_{11} = 2, \quad \mathrm{var}[X(t_2)] = C_{22} = 1\,.$$

However, stationarity of $X(t)$ requires $\mathrm{var}[X(t_1)] = \mathrm{var}[X(t_2)]$, which is a contradiction.

Stationary process

- **Consequence II** The 2D joint-PDF is shift invariant

$$
\begin{aligned}
f_{X(t_1),X(t_2)}(x_1,x_2) &= f_{X(t_1+\Delta t),X(t_2+\Delta t)}(x_1,x_2) \\
&= f_{X(0),X(t_2-t_1)}(x_1,x_2)
\end{aligned}
$$

$\Rightarrow$ only the "distance" τ between t_2 and t_1 matters.

$$
\begin{aligned}
R_X(t,\tau) &= R_X(\tau) \\
C_X(t,\tau) &= C_X(\tau) = R_X(\tau) - \mathrm{E}[X]^2
\end{aligned}
$$

Stationary process

Examples of stationary processes

- iid process; e.g. Bernoulli process
- Poisson process (random arrival process, ch. 13.4; we skip this)

Remember the PMF for the Bernoulli stochastic process:

$$P_{\boldsymbol{X}}(\boldsymbol{x}) = \prod_{i=1}^{k} p^{x_i}(1-p)^{1-x_i} = p^{x_1+\cdots+x_k}(1-p)^{k-(x_1+\cdots+x_k)}$$

which does not depend on the actual time.

Non-stationary processes are difficult to model and to handle in practice.

Problem 13.8.4

Let $X(t)$ be a stationary continuous-time random process. By sampling $X(t)$ every Δ seconds, we obtain the discrete-time random sequence $Y_n = X(n\Delta)$. Is Y_n a stationary sequence?

Problem 13.8.4

Let $X(t)$ be a stationary continuous-time random process. By sampling $X(t)$ every Δ seconds, we obtain the discrete-time random sequence $Y_n = X(n\Delta)$. Is Y_n a stationary sequence?

- Since $Y_{n_i+k} = X((n_i + k)\Delta)$ for a set of time samples $n_1, \cdots, n_m$

$$f_{Y_{n_1+k},\cdots,Y_{n_m+k}}(y_1, \cdots, y_m) = f_{X((n_1+k)\Delta),\cdots,X((n_m+k)\Delta)}(y_1, \cdots, y_m) .$$

- Since $X(t)$ is a stationary process,

$$f_{X((n_1+k)\Delta),\cdots,X((n_m+k)\Delta)}(y_1, \cdots, y_m) = f_{X(n_1\Delta),\cdots,X(n_m\Delta)}(y_1, \cdots, y_m) .$$

- Since $X(n_i\Delta) = Y_{n_i}$, we see that

$$f_{Y_{n_1+k},\cdots,Y_{n_m+k}}(y_1, \cdots, y_m) = f_{Y_{n_1},\cdots,Y_{n_m}}(y_1, \cdots, y_m)$$

Hence, Y_n is a stationary sequence.

Wide-Sense Stationary (WSS) Processes

- To show that a process is stationary, we need the overall joint-PDF.
  - Quite impossible to get, except for special cases.
- However, we can often estimate the process'
  - Expected value
  - Autocorrelation function
- If only the expected value and the autocorrelation function satisfy the property of stationarity, we call this process *wide sense stationary (WSS)*.
  - Hence, we don't know anything about other properties of the process!

Wide-Sense Stationary (WSS) Processes

- A process is **wide-sense stationary**, if and only if
  - The expected value $\mathrm{E}[X(t)]$ does not depend on time: $\mathrm{E}[X(t)] = c$.
  - The autocorrelation function only depends on the time difference τ and not the absolute time t:

$$R_X(t,\tau) = R_X(\tau)$$

or

$$R_X[n,k] = R_X(k)$$

- **Example:** sinusoidal random process, $X(t) = \sin(\omega t + \Phi)$ where Φ is uniformly distributed on $[0, 2\pi]$: derive that

$$C_X(\tau) = R_X(\tau) = \frac{1}{2}\cos(\omega\tau)$$

Problem 13.9.3 (important for later!)

True or False: If X_n is a wide sense stationary random sequence with $E[X_n] = c$, then $Y_n = X_n - X_{n-1}$ is a wide sense stationary random sequence.

Problem 13.9.3 (important for later!)

True or False: If X_n is a wide sense stationary random sequence with $\mathrm{E}[X_n] = c$, then $Y_n = X_n - X_{n-1}$ is a wide sense stationary random sequence.

True: First we observe that $\mathrm{E}[Y_n] = \mathrm{E}[X_n] - \mathrm{E}[X_{n-1}] = 0$, which does not depend on n. Second, we verify that

$$\begin{aligned} R_Y[n,k] &= \mathrm{E}[Y_n Y_{n+k}] \\ &= \mathrm{E}[(X_n - X_{n-1})(X_{n+k} - X_{n+k-1})] \\ &= \mathrm{E}[X_n X_{n+k}] - \mathrm{E}[X_n X_{n+k-1}] - \mathrm{E}[X_{n-1} X_{n+k}] + \mathrm{E}[X_{n-1} X_{n+k-1}] \\ &= R_X[k] - R_X[k-1] - R_X[k+1] + R_X[k], \end{aligned}$$

which does not depend on n. Hence, Y_n is WSS.

Problem 13.9.6

$X(t)$ and $Y(t)$ are independent wide sense stationary processes. Determine if $W(t) = X(t)Y(t)$ is wide-sense stationary.

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$X(t)$ and $Y(t)$ are independent wide sense stationary processes. Determine if $W(t) = X(t)Y(t)$ is wide-sense stationary.

True: Independence of $X(t)$ and $Y(t)$ implies

$$\mathrm{E}[W(t)] = \mathrm{E}[X(t)Y(t)] = \mathrm{E}[X(t)]\,\mathrm{E}[Y(t)] = \mu_X\,\mu_Y$$

and

$$\begin{aligned} R_W(t,\tau) &= \mathrm{E}[W(t)W(t+\tau)] \\ &= \mathrm{E}[X(t)Y(t)\,X(t+\tau)Y(t+\tau)] \\ &= \mathrm{E}[X(t)X(t+\tau)\,Y(t)Y(t+\tau)] \\ &= \mathrm{E}[X(t)X(t+\tau)]\,\mathrm{E}[Y(t)Y(t+\tau)] \quad \text{(by independence)} \\ &= R_X(\tau)R_Y(\tau) \end{aligned}$$

Since $W(t)$ has constant expected value and the autocorrelation depends only on the time difference τ, $W(t)$ is wide-sense stationary.

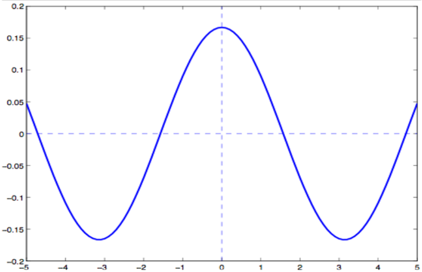
WSS Processes and the autocorrelation function

Important properties of $R_X(\tau)$ for WSS processes:

$$
\begin{aligned}
R_X(0) &\geq 0 \\
R_X(\tau) &= R_X(-\tau) \\
R_X(0) &\geq |R_X(\tau)|
\end{aligned}
$$

Example:

$$R_X(t,\tau) = \frac{1}{6}\cos(\omega\tau)$$



Cross correlation for Stochastic Processes

In addition to the autocorrelation function, we can also define the cross-correlation between two stochastic processes:

$$\begin{aligned} R_{XY}(t,\tau) &= \mathrm{E}[X(t)Y(t+\tau)] \\ R_{XY}[n,k] &= \mathrm{E}[X_n Y_{n+k}] \end{aligned}$$

- Two random processes $X(t)$ and $Y(t)$ are **jointly wide sense stationary**, if $X(t)$ and $Y(t)$ are wide sense stationary *and*

  $$R_{XY}(t,\tau) = R_{XY}(\tau).$$

- If $X(t)$ and $Y(t)$ are jointly WSS, then

  $$R_{XY}(\tau) = R_{YX}(-\tau).$$

  (and of course similar for time-discrete processes)

Example

X_n is a zero mean WSS stochastic process. Let $Y_n = (-1)^n X_n$.

- $\mathrm{E}[Y_n] = (-1)^n \mathrm{E}[X_n] = 0$, since X_n is zero mean.
- $$\begin{aligned} R_Y[n,k] &= \mathrm{E}[Y_n Y_{n+k}] \\ &= \mathrm{E}[(-1)^n X_n (-1)^{n+k} X_{n+k}] \\ &= (-1)^{2n+k} \mathrm{E}[X_n X_{n+k}] \;=\; (-1)^k R_X[k] \end{aligned}$$

  Process Y_n is WSS as $R_Y[n,k]$ only depends on k.

- $$\begin{aligned} R_{XY}[n,k] &= \mathrm{E}[X_n Y_{n+k}] \\ &= \mathrm{E}[X_n (-1)^{n+k} X_{n+k}] \\ &= (-1)^{n+k} \mathrm{E}[X_n X_{n+k}] \;=\; (-1)^{n+k} R_X[k] \end{aligned}$$

  X_n and Y_n are not jointly WSS as their cross-correlation function depends on both n and k.

Problem 13.10.2

$X(t)$ is a wide sense stationary random process. Let

$$(a) \quad Y(t) = X(t + a)$$

Are $Y(t)$ and $X(t)$ jointly wide sense stationary?

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$X(t)$ is a wide sense stationary random process. Let

$$(a) \quad Y(t) = X(t+a)$$

Are $Y(t)$ and $X(t)$ jointly wide sense stationary?

Since $\mathrm{E}[Y(t)] = \mathrm{E}[X(t+a)] = \mu_X$ and

$$\begin{aligned} R_Y(t,\tau) &= \mathrm{E}[Y(t)Y(t+\tau)] \\ &= \mathrm{E}[X(t+a)X(t+\tau+a)] = R_X(\tau), \end{aligned}$$

we have verified that $Y(t)$ is wide sense stationary. Next we calculate the cross correlation:

$$\begin{aligned} R_{XY}(t,\tau) &= \mathrm{E}[X(t)Y(t+\tau)] \\ &= \mathrm{E}[X(t)X(t+\tau+a)] = R_X(\tau+a). \end{aligned}$$

Since $R_{XY}(t,\tau)$ depends on the time difference τ but not on the absolute time t, we conclude that $X(t)$ and $Y(t)$ are jointly wide sense stationary.

Problem 13.10.2 (cont'd)

Now repeat for

$$(b) \quad Y(t) = X(at)$$

Problem 13.10.2 (cont'd)

Now repeat for

$$(b) \quad Y(t) = X(at)$$

Since $\mathrm{E}[Y(t) = \mathrm{E}[X(at)] = \mu_X$ and

$$\begin{aligned} R_Y(t,\tau) &= \mathrm{E}[Y(t)Y(t+\tau)] \\ &= \mathrm{E}[X(at)X(a(t+\tau))] \\ &= \mathrm{E}[X(at)X(at+a\tau)] = R_X(a\tau)\,, \end{aligned}$$

we have verified that $Y(t)$ is wide sense stationary. Now we calculate the cross correlation:

$$\begin{aligned} R_{XY}(t,\tau) &= \mathrm{E}[X(t)Y(t+\tau)] \\ &= \mathrm{E}[X(t)X(a(t+\tau))] \;=\; R_X((a-1)t+a\tau)\,. \end{aligned}$$

Except for the trivial case where $a = 1$ and $Y(t) = X(t)$, $R_{XY}(t,\tau)$ depends on both the absolute time t and the time difference τ: we conclude that $X(t)$ and $Y(t)$ are *not* jointly wide sense stationary.

To do for this lecture:

- Make some selected exercises:
  13.1.1, 13.3.1, 13.7.1, 13.7.3, 13.7.5, 13.9.3, 13.9.5, 13.9.7, 13.10.1, 13.10.3

Next lecture, we'll wrap up Ch.13 (ergodicity), and start with Supplement Sections 1 and 2.