Signal Processing EE2S31

Digital Signal Processing
Lecture 9: Exercise session

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Outline

- Multirate systems
  - Recap and finish
  - Exam question resit 2021
- FFT
  - Exercise 8.8
  - Decimation in frequency
  - Solution
- Sigma-delta modulators
  - 6.18
  - 6.20
- Quantization
  - Exam question final 2021

Overview and applications

Multirate systems

Multirate systems use different sampling rates at different stages of the digital processing.

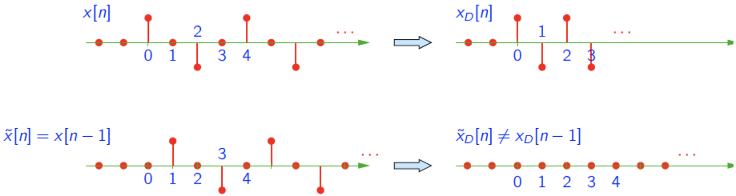
Examples:

- telecommunication systems (speech, video)
- digital antialiasing (CD players)
- subband coding for speech and audio
- multimodal biomedical monitoring (EEG, ECG)

Decimation by a factor M

$$T_s' = MT \Rightarrow x_D[n] = x_a(nT_s') = x_a(nMT_s) = x[nM]$$

Decimation is a linear time-varying operation!

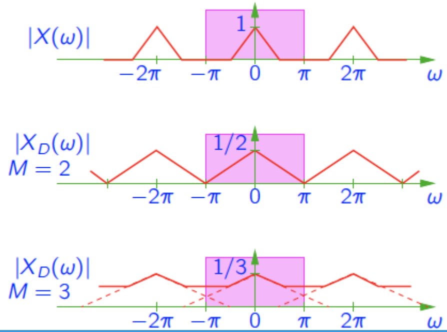


Decimation by a factor M

Spectrum of the decimated signal:

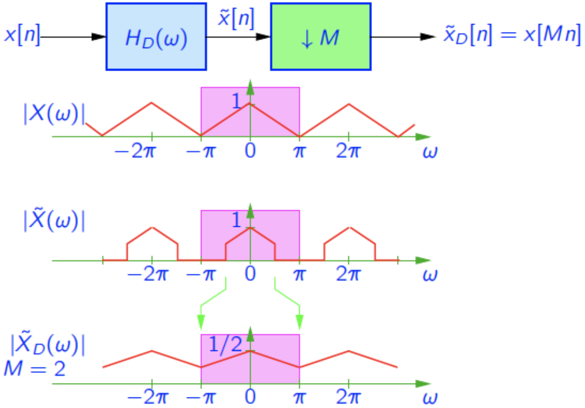
$$X_D(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(z^{1/M}e^{-j2\pi k/M}), \qquad X_D(\omega) = \frac{1}{M}\sum_{k=0}^{M-1} X(\frac{\omega - 2\pi k}{M})$$

The spectrum of the decimated signal $X_D(\omega)$ consists of shifted and stretched copies of the original spectrum $X(\omega)$. (Note, $x[n]$ is digital, so $X(\omega)$ is already periodic!)



Decimantion by a factor M

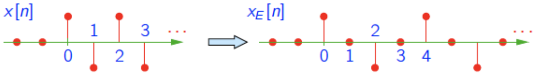
How to avoid aliasing?



We need to use an anti-aliasing decimationfilter (low-pass filter) with cut-off at $\frac{\pi}{M}$.

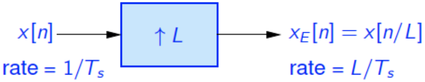
Interpolation with a factor L

Interpolation (=upsampling) with a factor L means to insert $L-1$ zeros:



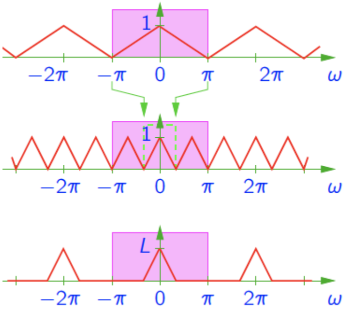
$$x_E[n] = \begin{cases} x[k], & n = kL \\ 0, & \text{otherwise} \end{cases}$$

The sampling rate of the signal increases by a factor L. There is no information loss!



Interpolation with a factor L

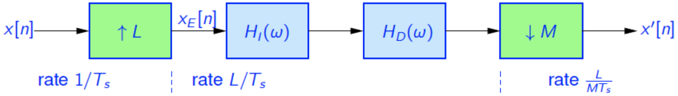
$$X_E(z) = X(z^L),\ X_E(\omega) = X(\omega L)$$



- The spectrum of the signal contracts with a factor L, and $L-1$ copies occur in the fundamental interval.
- We can remove the copies using a digital low-pass filter
- the overall system (i.e. sampling with $1/T_s$, upsampling and filtering) is then equivalent with a system sampled at a rate of L/T_s.

Sampling rate conversion by factor L/M

We can combine upsampling with L and downsampling with M to implement a sampling rate conversion with any rational factor L/M.

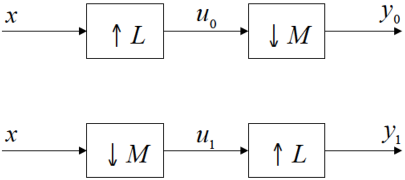


The two low-pass filters can be combined in one with a cut-off of $\omega_c = min(\pi/L, \pi/M)$.

Can we do the other way around, first decimate then interpolate? (Would that give the same output signal?)

Sampling rate conversion by factor L/M

Exercise 11.9:



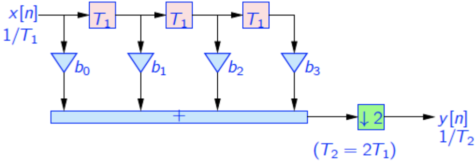
General rule: $y_0 = y_1$ if and only if L and M are relative primes!

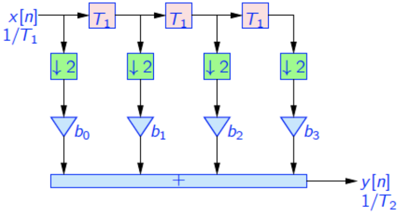
Implementation of sampling rate conversion

Goal

Efficiently implement multirate conversion systems

Example:





Note: In general, we cannot swap a filter and a sampling rate converter!

Implementation of sampling rate conversion

Objective

Let's modify the filter realisation such that filters and sampling rate converters can be swapped.

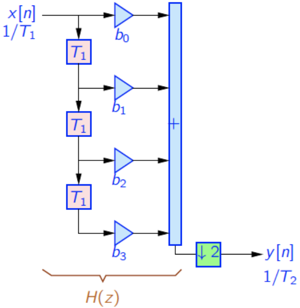
Outcome

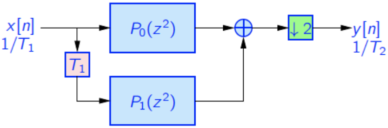
The resulting circuit is overall more efficient

Two ingredients:

- Polyphase filter structures
- Noble identities

Polyphase filter structures





$P_0(z) = b_0 + b_2 z^{-1}$
$P_1(z) = b_1 + b_3 z^{-1}$

$$\begin{aligned} H(z) &= b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} = \\ &= (b_0 + b_2 z^{-2}) + z^{-1}(b_1 + b_3 z^{-2}) = \\ &= P_0(z^2) + z^{-1} P_1(z^2) \end{aligned}$$

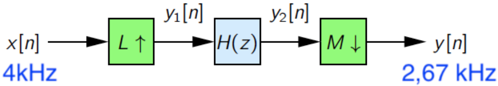
Noble identities

Identity 1:

Identity 2:

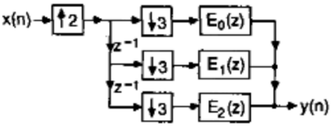
Example

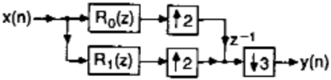
How to efficiently implement the following system:



Example

Step 1 (2 possibilities):



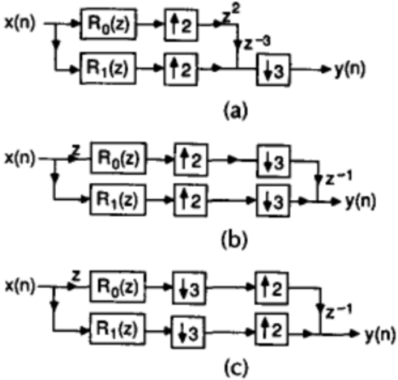


$(R_k(z) = E_{M-1-k}(z))$

Which one is more efficient (in terms of the rate at which the filters operate?)

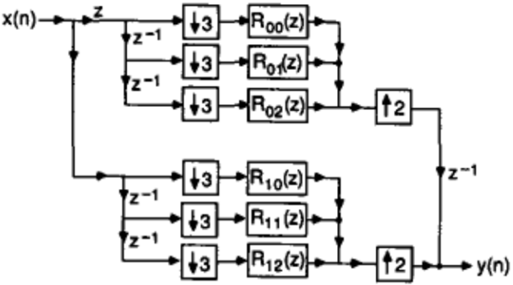
Example

Step 2 (let us work further on option 2):



Example

Most efficient implementation:

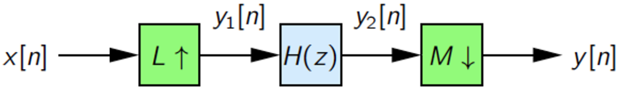


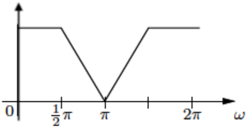
Now filters run at 4/3 kHz.

- Overview and applications
- Sampling rate conversion
  - Decimation by factor M
  - Upsampling by a factor L
  - Sampling rate conversion by factor L/M
- Implementation of sampling rate conversion
  - Sampling rate conversion using polyphase filters
  - ~~Multistage Sampling rate conversion~~

Exercise: Resit 2021 Q6

Given a multirate conversion system with a block scheme shown below with L=2 and M=5. The sampling rate at the input is 100Hz. The amplitude spectrum $|X(\omega)|$ of the input signal $x[n]$ is also shown.

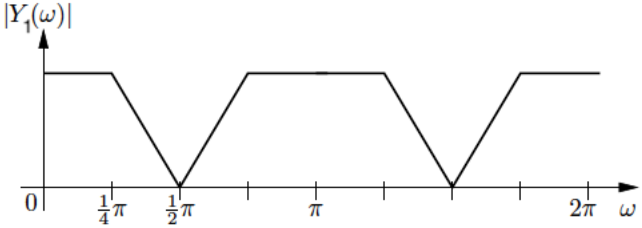




Question 6.a

(a) Give a formula for $Y_1(\omega)$ in terms of $X(\omega)$ and draw a graphic for the amplitude spectrum $|Y_1(\omega)|$!

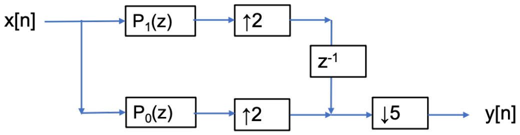
A: $Y_1(\omega) = X(2\omega)$ (Attention: typo on website!)
The amplitude spectrum is shown below:



Figure

Question 6.b

(b) Let us consider the implementation of the conversion system shown in Fig. 2.



Figure

What is the role of the filters $P_i(z)$ and how are they related to $H(z)$?

A: $P_i(z)$ are the polyphase representation of the filter $H(z)$, that is $H(z) = P_0(z^2) + z^{-1}P_1(z^2)$.

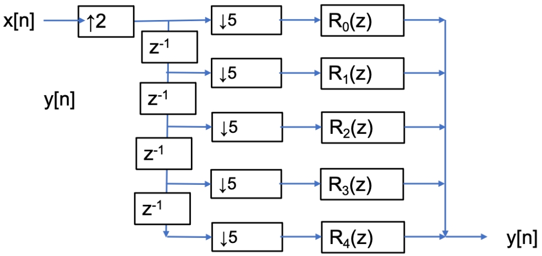
(c) At which rate do the filters operate in this implementation?

A: In the given implementation the filter operates at the same rate as the input, i.e. 100Hz.

Questions 6.c,d

(d) Draw an alternative, more efficient implementation of the multirate conversion system (in terms of the rate at which the filters operate!) At which rate do the filters operate now?

A: In the implementation below the filters operate at 40Hz (obtained from the original block diagram by replacing H(z) with a 5-component polyphase filter and exchanging the order of the filter and the downsampler):



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Exercise 8.8

8.8 Compute the eight-point DFT of the sequence

$$x(n) = \begin{cases} 1, & 0 \le n \le 7 \\ 0, & \text{otherwise} \end{cases}$$

by using the decimation-in-frequency FFT algorithm described in the text.

Decimation in Frequency (1)

Idea: split the regular DFT formula into two part, summing the first and second half of the terms separately:

$$
\begin{aligned}
X(k) &= \sum_{n=0}^{(N/2)-1} x(n) W_N^{kn} + \sum_{n=N/2}^{N-1} x(n) W_N^{kn} \\
&= \sum_{n=0}^{(N/2)-1} x(n) W_N^{kn} + W_N^{Nk/2} \sum_{n=0}^{(N/2)-1} x\left(n + \frac{N}{2}\right) W_N^{kn}
\end{aligned}
\tag{8.1.33}
$$

Since $W_N^{kN/2} = (-1)^k$, the expression (8.1.33) can be rewritten as

$$
X(k) = \sum_{n=0}^{(N/2)-1} \left[x(n) + (-1)^k x\left(n + \frac{N}{2}\right) \right] W_N^{kn} \tag{8.1.34}
$$

Now, let us split (decimate) $X(k)$ into the even- and odd-numbered samples. Thus we obtain

$$
X(2k) = \sum_{n=0}^{(N/2)-1} \left[x(n) + x\left(n + \frac{N}{2}\right) \right] W_{N/2}^{kn}, \qquad k = 0, 1, \ldots, \frac{N}{2} - 1 \tag{8.1.35}
$$

and

$$
X(2k+1) = \sum_{n=0}^{(N/2)-1} \left\{ \left[x(n) - x\left(n + \frac{N}{2}\right) \right] W_N^{n} \right\} W_{N/2}^{kn}, \qquad k = 0, 1, \ldots, \frac{N}{2} - 1 \tag{8.1.36}
$$

Decimation in Frequency (2)

$$g_1(n) = x(n) + x\left(n + \frac{N}{2}\right)$$
$$g_2(n) = \left[x(n) - x\left(n + \frac{N}{2}\right)\right] W_N^n, \qquad n = 0, 1, 2, \ldots, \frac{N}{2} - 1 \tag{8.1.37}$$

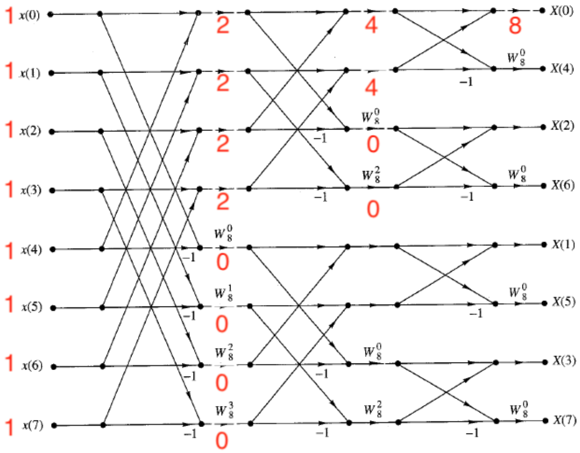
then

$$X(2k) = \sum_{n=0}^{(N/2)-1} g_1(n) W_{N/2}^{kn}$$
$$X(2k+1) = \sum_{n=0}^{(N/2)-1} g_2(n) W_{N/2}^{kn} \tag{8.1.38}$$

In words, the even coefficients of the N-point DFT of a given sequence can be obtained as the N/2 DFT of a sequence obtained by summing the first and second half of the original sequence. A similar statement can be made for the odd coefficient (making a subtraction instead of a summation of the half sequences; followed by a multiplication with W_N^n)

Exercise 8.8 solution

For an 8-point DFT, a three-stage decimation-in-frequency procedure leads to algorithm below, depicted on a butterfly diagram:
Use the butterlfy diagram below to perform the necessary computations!



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Exercise 6.18

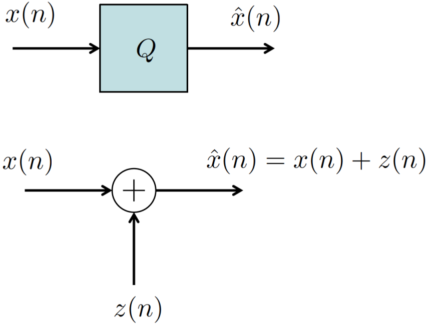
6.18 Let $x_a(t)$ be a bandlimited signal with fixed bandwidth B and variance σ_x^2.

(a) Show that the signal-to-quantization noise ratio, $\mathrm{SQNR} = 10\log_{10}(\sigma_x^2/\sigma_e^2)$, increases by 3 dB each time we double the sampling frequency F_s. Assume that the quantization noise model discussed in Section 6.3.3 is valid.

(b) If we wish to increase the SQNR of a quantizer by doubling its sampling frequency, what is the most efficient way to do it? Should we choose a linear multibit A/D converter or an oversampling one?

Quantization error

Mathematical model of quantization:



Assumptions:

- input signal $x[n]$ is the realizatin of a zero-mean WSS process
- quantization noise is white
- quantization niose is uncorrelated to the input

PSD of white noise

Example 1

Let $X(t)$ be white noise: WSS, zero mean, with $R_X(\tau) = N_0\delta(\tau)$.
Determine $S_X(f)$.

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} \mathrm{d}\tau = N_0 \int_{-\infty}^{\infty} \delta(\tau) e^{-j2\pi f\tau} \mathrm{d}\tau = N_0$$

The PSD is flat, all frequencies are equally strong, hence "white" (cf. white light)

Power of white noise

Example 2

Given is a WSS process $X(t)$ with

$$R_X(\tau) = Ae^{-b|\tau|}, \quad b > 0.$$

Derive the PSD $S_X(f)$ and calculate the average power $\mathrm{E}[X^2(t)]$.

Using Table 1 (Fourier transform pairs), we learn that

$$ae^{-a|\tau|} \Leftrightarrow \frac{2a^2}{a^2 + (2\pi f)^2}$$

$$S_X(f) = \frac{A}{b}\frac{2b^2}{b^2 + (2\pi f)^2} = \frac{2Ab}{b^2 + (2\pi f)^2}$$

The average power is:

$$\begin{aligned} \mathrm{E}[X^2(t)] &= R_X(0) = A \\ &= \int_{-\infty}^{\infty} S_X(f)\mathrm{d}f = \left[\frac{A}{\pi}\arctan(2\pi f/b)\right]_{-\infty}^{\infty} \end{aligned}$$

Exercise 6.18

To calucate the SQNR, we have to calculate the power of the noise within the signal band.
The width of signal band (in normalized frequencies) depends on the ratio of the badnwidth and the sampling rate!

$$\begin{aligned} P_n &= \int_{-\frac{B}{F_s}}^{\frac{B}{F_s}} P_d df \\ &= \frac{2B}{F_s} P_d \\ &= \sigma_e^2 \\ \text{SQNR} &= 10 log_{10} \frac{\sigma_x^2}{\sigma_e^2} \\ &= 10 log_{10} \frac{\sigma_x^2 F_s}{2BP_d} \end{aligned}$$

$$= 10 \log_{10} \frac{\sigma_x^2}{2BP_d} + 10 \log_{10} F_s$$

$$SQNR_2 = \log_{10} \frac{\sigma_x^2}{2BP_d} + 10 \log_{10} 2F_s = \log_{10} \frac{\sigma_x^2}{2BP_d} + 10 \log_{10} F_s + 10 \log_{10} 2 = \log_{10} \frac{\sigma_x^2}{2BP_d} + 10 \log_{10} F_s + 3$$

Exercise 6.20

6.20 Consider the second-order SDM model shown in Fig. P6.20.

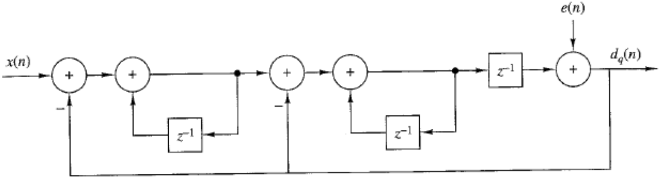


Figure P6.20

(a) Determine the signal and noise system functions $H_s(z)$ and $H_n(z)$, respectively.

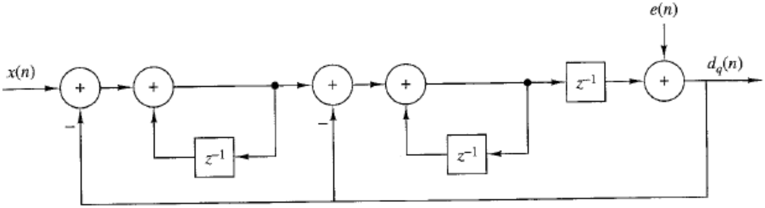
(b) Plot the magnitude response for the noise system function and compare it with the one for the first-order SDM. Can you explain the 6-dB difference from these curves?

(c) Show that the in-band quantization noise power σ_n^2 is given approximately by

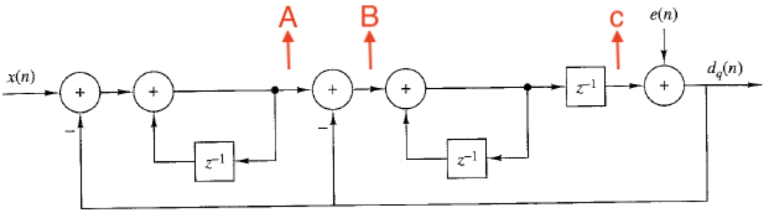
$$\sigma_n^2 \approx \frac{\pi \sigma_e^2}{5} \left(\frac{2B}{F_s} \right)^5$$

which implies a 15-dB increase for every doubling of the sampling frequency.

6.20 (a)



6.20 (a)



$$H(z) = \frac{1}{1 - z^{-1}}$$

$$(X(z) - D(z))H(z) \qquad \text{(A)}$$

$$(X(z) - D(z))H(z) - D(z) \qquad \text{(B)}$$

$$[(X(z) - D(z))H(z) - D(z)]H(z)z^{-1} \qquad \text{(C)}$$

$$[(X(z) - D(z))H(z) - D(z)]H(z)z^{-1} + E(z) = D(z)$$

6.20 (a)

$$\left[(X(z) - D(z))H(z) - D(z)\right]H(z)z^{-1} + E(z) = D(z)$$

$$\left[(X(z) - D(z))\frac{1}{1-z^{-1}} - D(z)\right]\frac{1}{1-z^{-1}}z^{-1} - D(z) = E(z)$$

$$X(z)\frac{1}{1-z^{-1}}\frac{z^{-1}}{1-z^{-1}} - D(z)\frac{1}{1-z^{-1}}\frac{z^{-1}}{1-z^{-1}} - D(z)\frac{z^{-1}}{1-z^{-1}} - D(z) = E(z)$$

$$X(z)\frac{z^{-1}}{(1-z^{-1})^2} - D(z)\frac{1}{(1-z^{-1})^2} = E(z)$$

$$H_s(z) = \frac{\frac{1}{(1-z^{-1})^2}}{\frac{z^{-1}}{(1-z^{-1})^2}} = z^{-1}$$

$$H_n(z) = \frac{1}{\frac{1}{(1-z^{-1})^2}} = (1-z^{-1})^2$$

6.20 (b)

Question: plot and compare the magnitude of the frequency reponse of the 1st and 2nd order system: $H_{n1}(z) = (1 - z^{-1})$ and $H_{n2}(z) = (1 - z^{-1})^2$

Answer:

- frequency response: $z = e^{j\omega}$
- magnitude: $|H(\omega)|$

$$|H_{n1}(\omega)| = (1 - e^{-j\omega}) = |1 - \cos\omega + j\sin\omega| = \sqrt{(1-\cos\omega)^2 + \sin^2\omega} =$$

$$= \sqrt{(1 + \cos^2\omega - 2\cos\omega + \sin^2\omega} = \sqrt{2 - 2\cos\omega}$$

$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x \rightarrow 1 - \cos(2x) = 2\sin^2 x$

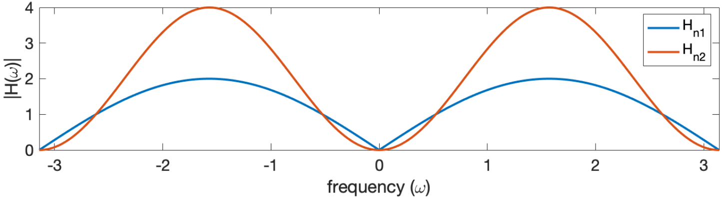
$$|H_{n1}(\omega)| = \sqrt{2 - 2\cos\omega} = \sqrt{4\sin^2\frac{\omega}{2}} = 2|\sin\frac{\omega}{2}| \rightarrow 2|\sin\frac{\pi F}{F_s}|$$

$$|H_{n2}(\omega)| = 4\sin^2\frac{\pi F}{F_s}$$

6.20 (b)

$$|H_{n1}(\omega)| = \sqrt{2-2\cos\omega} = \sqrt{4\sin^2\frac{\omega}{2}} = 2|\sin\frac{\omega}{2}| \rightarrow 2|\sin\frac{\pi F}{F_s}|$$

$$|H_{n2}(\omega)| = 4\sin^2\frac{\pi F}{F_s}$$



$$\sigma_n^2 = \int_{-B}^{B} |H_n(F)|^2 S_e(F) dF \text{ (book 6.6.11)} \rightarrow$$

$\rightarrow$ factor of 4 reduction if $B << F_s \rightarrow$ 6dB

6.20 (c)

$$\sigma_n^2 = \int_{-B}^{B} |H_n(F)|^2 S_e(F) dF$$

$$= \int_{-B}^{B} |H_n(F)|^2 \frac{\sigma_e^2}{F_s} dF = 2\int_{0}^{B} (4(\sin\frac{\pi F}{F_s})^2)^2 \frac{\sigma_e^2}{F_s} dF$$

for small x $\sin x \approx x$

$$2\int_{0}^{B} (4(\frac{\pi F}{F_s})^2)^2 \frac{\sigma_e^2}{F_s} dF = 32\frac{\pi^4\sigma_e^2}{F_s^5}\int_{0}^{B} F^4 dF =$$

$$32\frac{\pi^4\sigma_e^2}{F_s^5}\frac{1}{5}(B^5) = \frac{\pi^4\sigma_e^2}{5}(\frac{2B}{F_s})^5$$

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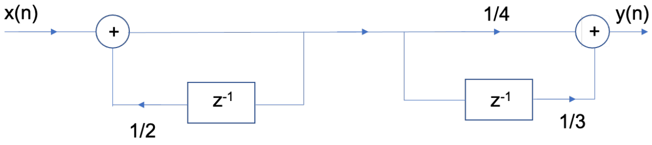
- **Quantization**
  - Exam question final 2021

Final exam 2021 Question 4

Let us consider a first-order IIR filter with impulse response

$$h(n) = \frac{1}{4}\left(\frac{1}{2}\right)^{n} u[n] + \frac{1}{3}\left(\frac{1}{2}\right)^{(n-1)} u[n-1]$$

A direct form I realization of the filter is shown below.



The outputs of the multipliers in this system are quantized using a midtread quantizer and a sign-magnitude coding scheme with 3 bits plus the sign bit. The quantizer can encode values between $(-1, 1)$.
We model the effect of quantization as an additive noise source $e(n)$, and we assume that $e(n)$ is an uncorrelated wide-sense stationary process that is uniformly distributed.

Question 4a

(a) What is the variance of the quantization noise of this particular quantizer?

A: $\sigma_E^2 = \frac{\Delta^2}{12}$, where Δ is the step size of the quantizer, that is $\Delta = \frac{R}{2^{b+1}} = \frac{2}{2^{3+1}} = 0.125$ with R the range of the quantizer and b is the number of bits. So, $\sigma_E^2 = \frac{0.125^2}{12} = 0.0013$.

Question 4b

(b) Now let us consider the quantization noise at the output of the filter. Compute the variance of the quantization noise at the output of the filter!

A: For a given quantization noise source $\sigma_Q^2 = \sigma_E^2 \sum_{n=-\infty}^{\infty} |h[n]|^2$, where $h[n]$ is the impulse response of part of the system that the noise passes through. The total noise variance is the sum of the output variance of all contributing noise sources. In our case, the first noise source $e1$ (of multiplier $a_1 = \frac{1}{2}$) passes through the whole system, while the other two ($e1$ and $e2$, corresponding to $b_0 = \frac{1}{4}$ and $b_1 = \frac{1}{3}$) appear directly at the output. Therefore,

$$\sigma_{Qtotal}^2 = \sigma_{E1}^2 \sum_{n=-\infty}^{\infty} |h[n]|^2 + \sigma_{E2}^2 + \sigma_{E3}^2$$

Question 4b (continued)

Let us write the impulse respone of the system in terms of the variables for the clarity of the derivation: $h[n] = b_0 a_1^n u[n] + b_1 a_1^{(n-1)} u[n-1]$
For $m = 0$, the impulse response is $h(0) = b_0$.
For $m \geq 1$, sample m of the impulse reponse can be written as
$h[m] = b_0 a_1 (a_1)^{m-1} + b_1 (a_1)^{m-1}$ Therefore,

$$\sum_{n=-\infty}^{\infty} |h[n]|^2 = b_0^2 + \sum_{m=1}^{\infty} (a_1 b_0 a_1^{m-1} + b_1 a_1^{m-1})^2 = \sum_{n=0}^{\infty} (a_1 b_0 a_1^n + b_1 a_1^n)^2$$

$$= b_0^2 + \sum_{n=0}^{\infty} a_1^2 b_0^2 a_1^{2n} + b_1^2 a_1^{2n} + a_1 b_0 b_1 a_1^{2n}$$

$$= b_0^2 + \frac{1}{1 - a_1^2} (a_1^2 b_0^2 + b_1^2 + 2 a_1 b_0 b_1) = 0.34$$

Substituting back to the previous equation and using the answer to (a):

$$\sigma^2_{Qtotal} = 0.0013 \cdot (2 + 0.34) = 0.0030$$

Please fill in the evaluation