Signal Processing EE2S31

Digital Signal Processing
Lecture 7: Sigma-delta ADCs

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Outline

Strategies to improve A/D conversion

- Oversampling
- Differential quantization
- Discrete-time model of sigma-delta ADCs
- Noise shaping

Signal-to-quantization-noise ratio of the quantizer

$$SQNR = 10\log_{10}\frac{\sigma_x^2}{\sigma_z^2}, \text{ with } \sigma_z^2 = \frac{\Delta^2}{12} \text{ and } \Delta = \frac{R}{2^{B+1}}$$

- the range of the input (which is proportional to σ_x) should match the range of the quantizer R
- according to formulas above, σ_z is proportional to Δ, but Δ is proportional to R

$$\Downarrow$$

- for a given B, σ_z^2 is proportional to σ_x^2

$$\Downarrow$$

- if we aim for maintaining a certain SQNR, reducing the variance of the input signal to the quantizer reduces the number of necessary bits.

Strategies to improve SQNR:

- Differential quantization
- Oversampling

Differential quantization

Idea: to reduce the range of the input to the quantizer, let us quantize the differential signal!

$$d[n] = x[n] - x[n-1]$$

What is the variance of the differential signal?

$$\begin{aligned}
\sigma_d^2 &= E\left[d^2[n]\right] = E\left[x[n] - x[n-1]\right]^2 \\
&= E\left[x^2[n]\right] - 2E\left[x[n]x[n-1]\right] + E\left[x^2[n-1]\right] = \ldots = \\
&= 2\sigma_x^2[1 - r_x[1]], \text{ with } r_x[1] = R_x[1]/R_x[0]
\end{aligned}$$

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- differential quantization is advantageous if $r_x[1] > 0.5$
- high autocorrelation at lag 1 (i.e. high $r_x(1)$) can be achieved by oversampling (= sampling well above Nyquist)!
- Even better alternative: prediction!

Differential predictive quantizer

Let us now quantize the following quantity instead:

$$d(n) = x(n) - ax(n-1)$$

What is the optimal choice of a? (exercise 6.16)

$$\begin{aligned}\sigma_d^2 &= E\left[d^2[n]\right] = E\left[x[n] - ax[n-1]\right]^2 \\ &= E\left[x^2[n]\right] - 2E\left[ax[n]x[n-1]\right] + E\left[a^2x^2[n-1]\right] = \\ &= (1+a^2)\sigma_x^2 - 2aR_x[1]\end{aligned}$$

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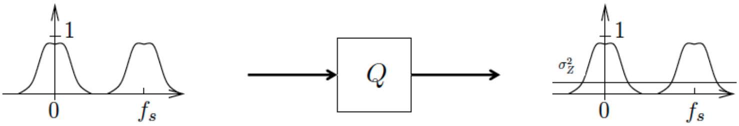
Then, by substituting back, we obtain $\sigma_d^2 = \sigma_x^2(1 - r_x^2(1)) \leq \sigma_x^2$ indicating that differental predictive quantization is always advantageous!

Sampling and quantization noise

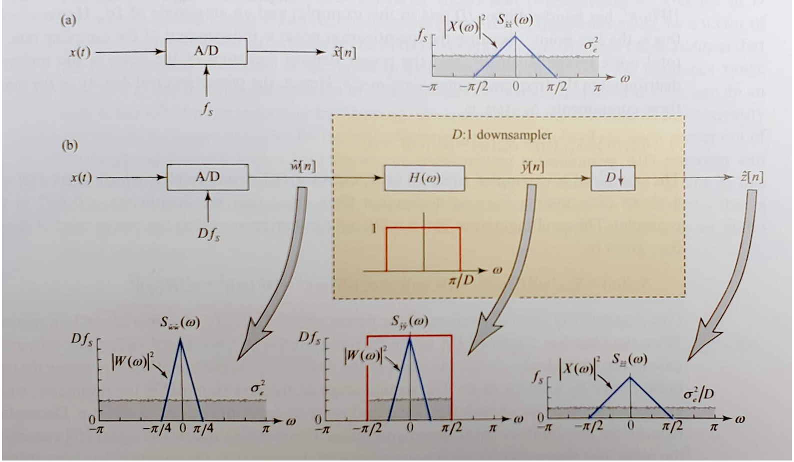
Recall that the spectrum of a sampled signal equals

$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a(F + kF_s)$$

The effect of quantizaion is an addition of white quantization noise with power σ_z^2.



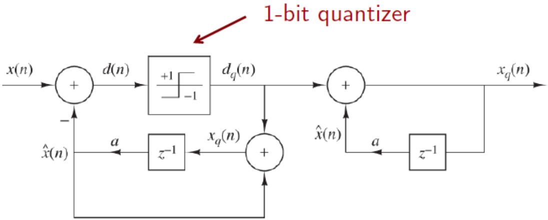
Oversampling and quantization noise



In case of oversampling by a factor of D (see (b) above), the signal to quantization noise power ration is increased by a factor D.

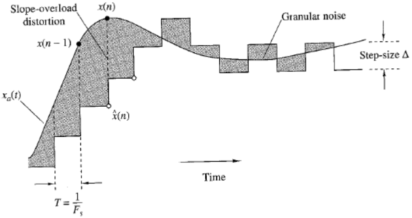
Sigma-delta modulator

Let us combine the two principles, i.e. differential quantization and oversampling!

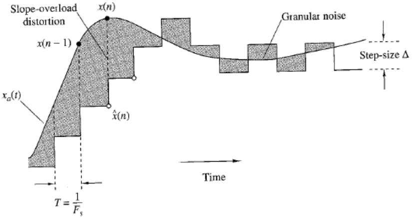


- coder consists of the predictor and the quantizer
- predictor provides an estimate $\hat{x}[n]$ of $x[n]$ using its past sample(s) in case of a first (or higher order) predictor
- with $a = 1$ the predictor is a simple accumulator
- quantizer quantizes the difference $d[n] = x[n] - \hat{x}[n]$
- oversampling allows the use of a 1-bit quantizer, providing a staircase approximation of the signal
- decoder reconstructs the signal from the differential quantized values

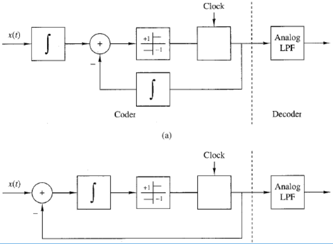
Sigma-delta modulator



Sigma-delta modulator



To reduce distortion, we can introduce an integrator before the differential quantizer. Uniting the two integrators yields a simpler circuit, where a single intergrator (*sigma*) is followed by a differentiaror (*delta*).



Sigma-delta modulator: discrete model

Let us model the SDM as a discrete system: the integrator as H(z) and the quantizer as an additive white noise!

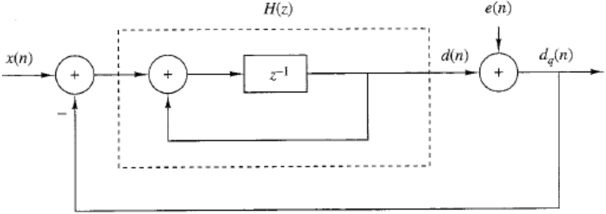


Figure 6.6.4 Discrete-time model of sigma-delta modulation

The z-transform of the sequence $d_q(n)$ can be separated into a response to the signal and a response to noise:

$$D_q(z) = \frac{H(z)}{1 + H(z)} X(z) + \frac{1}{1 + H(z)} E(z) =$$

$$= H_s(z) X(z) + H_n(z) E(z)$$

Noise shaping

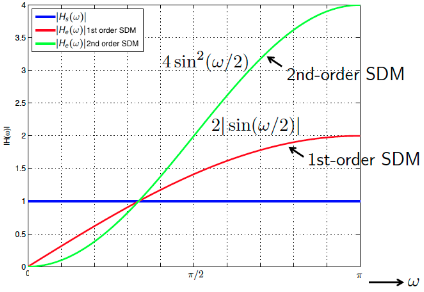
Let us work out the previous formula for $D_q(z)$, with $H(z) = \frac{z^{-1}}{1-z^{-1}}$

$$D_q(z) = \frac{H(z)}{1+H(z)}X(z) + \frac{1}{1+H(z)}E(z) = \dots$$
$$= z^{-1}X(z) + (1 - z^{-1})E(z)$$

Therefore,

$$|H_s(\omega)| = 1 \text{ and } |H_e(\omega)| = 2|sin(\omega/2)|$$

Noise-shaping



- In case of oversampling, signal is restricted to the lowest (normalized) frequencies, and noise power is spread over a relatively larger frequency band (over the full range between 0 and 2π)
- The signal is not distorted by the system H_s.
- The noise is attenuated at low frequencies.
- Therefore, higher SNR can be achieved by oversampling (see exercise 6.20)

Sigma-delta modulation

Summary

Oversampling A/D converters (=sigma-delta modulators) using 1-bit quantization can achieve very good accuracy at low cost, based on the principle of oversampling, differential quantization and noise shaping.