Frequency domain sampling: DFT

Borbala Hunyadi

Delft University of Technology, The Netherlands

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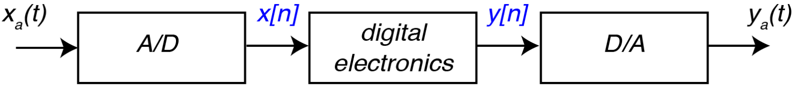
Recap: DSP

Digital Signal Processing

Processing of analog signals by means of discrete-time operations implemented on digital hardware

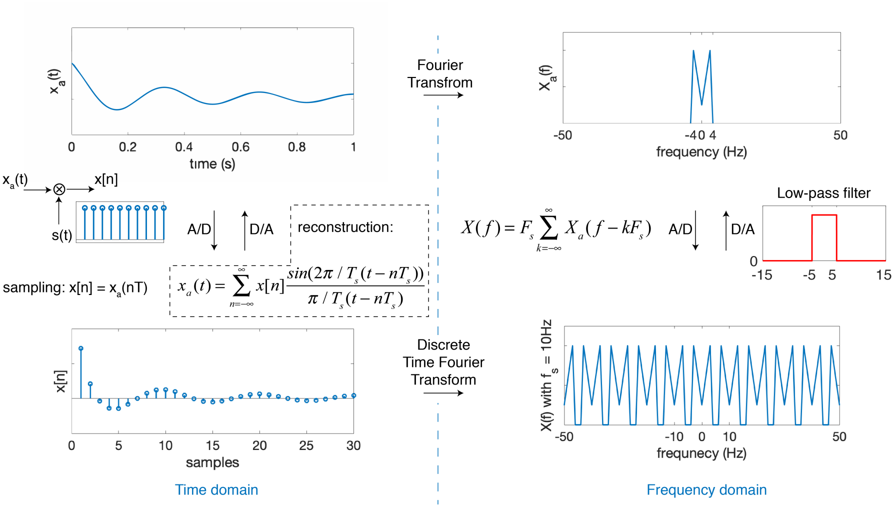


sampling
quantization

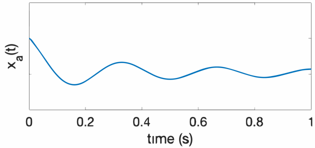
filtering
spectral analysis

reconstruction

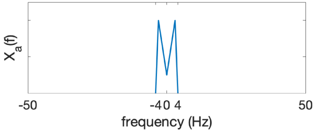
Recap: (Non-ideal) sampling and reconstruction

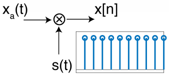


Recap: (Non-ideal) sampling and reconstruction



Fourier Transfrom →





A/D ↓ ↑ D/A

reconstruction:

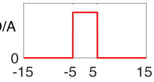
$$x_a(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin(2\pi / T_s (t - nT_s))}{\pi / T_s (t - nT_s)}$$

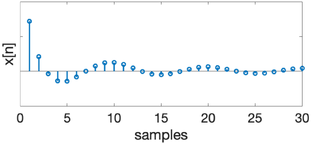
sampling: $x[n] = x_a(nT)$

$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a(f - kF_s)$$

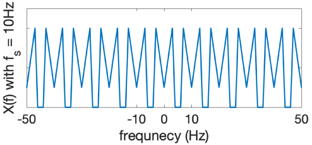
A/D ↓ ↑ D/A

Low-pass filter





Discrete Time Fourier Transform →



Time domain

Frequency domain

Spectrum $X(f)$ is a continuous function of frequency!

Frequency domain sampling

- $X(f)$, computed via the discrete time fourier transform (DTFT), is the spectrum of the sampled signal $x[n]$
- $X(f)$ is a continuous function of frequency
- $X(f)$ is not a convenient representation of $x[n]$ for a digital system

Frequency domain sampling

- $X(f)$, computed via the discrete time fourier transform (DTFT), is the spectrum of the sampled signal $x[n]$
- $X(f)$ is a continuous function of frequency
- $X(f)$ is not a convenient representation of $x[n]$ for a digital system

Goal of this lecture

We will consider the representation of x[n] using the samples of its spectrum X(f). This will lead to the definition of the discrete Fourier Transform (DFT).

Frequency domain sampling

Let's consider the spectrum of a discrete-time aperiodic signal $x[n]$! Its spectrum is

- continuous (why?)
- periodic (why?)

We want to represent the continuous spectrum with its samples in the frequency domain:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \rightarrow X[k\delta\omega]$$

Fundamental questions (analogous to time-domain sampling):

- How many samples do we need?
- How to reconstruct the continuous spectrum?

Frequency domain sampling

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Fundamental questions (analogous to time-domain sampling):

- **How many samples do we need?**
- How to reconstruct the continuous spectrum?

Frequency domain sampling

Because of the periodicity, we only need to sample in the range $0 \leq \omega \leq 2\pi$. Let's take N equidistant samples:

$$
\begin{aligned}
X(\omega) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \Rightarrow \\
X(\frac{2\pi}{N}k) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi kn/N}, \ k = 0, 1, \dots, N-1 \\
&= \ \dots + \sum_{n=-N}^{-1} x[n] e^{-j2\pi kn/N} + \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} + \sum_{N}^{2N-1} x[n] e^{-j2\pi kn/N} + \dots \\
&= \sum_{m=-\infty}^{\infty} \sum_{n=mN}^{mN+N-1} x[n] e^{-j2\pi kn/N} \\
&= \sum_{n=0}^{N-1} \sum_{m=-\infty}^{\infty} x[n-mN] e^{-j2\pi kn/N}
\end{aligned}
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Note the analogy with the derivation related to time-domain sampling in the previous lecture!

Frequency domain sampling

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\end{aligned}
$$

Let's define $x_p[n] = \sum_{m=-\infty}^{\infty} x[n-mN]$!

Frequency domain sampling

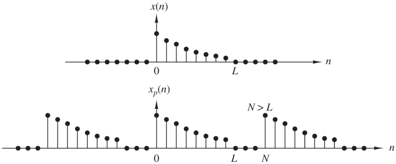
Because of the periodicity, we only need to sample in the range $0 \leq \omega \leq 2\pi$. Let's take N equidistant samples:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \Rightarrow$$

$$\begin{aligned}
X(\frac{2\pi}{N}k) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi kn/N}, \; k = 0, 1, ..., N-1 \\
&= \; ... + \sum_{n=-N}^{-1} x[n]e^{-j2\pi kn/N} + \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} + \sum_{N}^{2N-1} x[n]e^{-j2\pi kn/N} + ... \\
&= \sum_{m=-\infty}^{\infty} \sum_{n=mN}^{mN+N-1} x[n]e^{-j2\pi kn/N} \\
&= \sum_{n=0}^{N-1} \sum_{m=-\infty}^{\infty} x[n-mN]e^{-j2\pi kn/N} = \sum_{n=0}^{N-1} x_p[n]e^{-j2\pi kn/N}
\end{aligned}$$

Frequency-domain sampling

$$x_p[n] = \sum_{m=-\infty}^{\infty} x[n - mN]$$



Considering that $x_p[n]$ is periodic with period N, it can be expanded in a Fourier series as:

$$x_p[n] = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$$, with n=0, 1, ... N-1, where (synthesis)

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p[n] e^{-j2\pi kn/N}$$, with k=0, 1, ... N-1 (analysis)

Frequency-domain sampling

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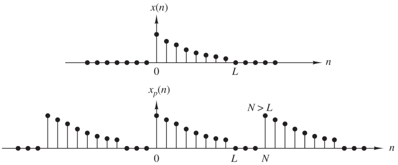

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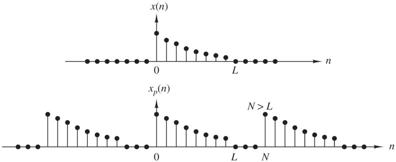
$$c_k = \frac{1}{N} X(\frac{2\pi}{N} k)$$ Result from the previous derivation!

Frequency-domain sampling

$$x_p[n] = \sum_{m=-\infty}^{\infty} x[n - mN]$$



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$$c_k = \frac{1}{N} \underbrace{\sum_{n=0}^{N-1} x_p[n] e^{-j2\pi kn/N}}$$, with k=0, 1, ... N-1 (analysis)

$$c_k = \frac{1}{N} X(\frac{2\pi}{N} k)$$ Let's substitute into the synthesis equation!

Reconstruction of a periodic signal

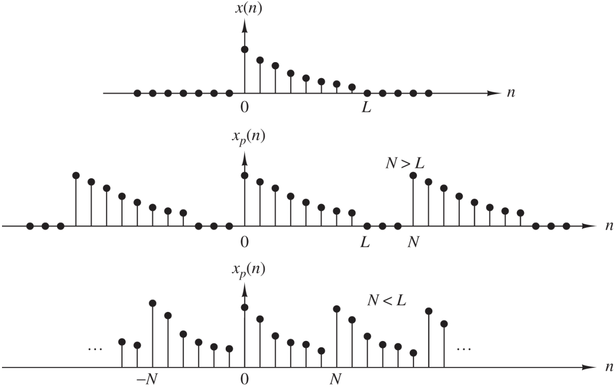
$$x_p[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(\frac{2\pi}{N}k)e^{j2\pi kn/N} \qquad , \text{n} = 0, 1, \ldots \text{N - 1}$$

This result provides the reconstruction formula of the signal $x_p[n]$ (i.e. periodic extension of the finite sequence $x[n]$) from the samples of its spectrum $X(\omega)$.

What about reconstrucing $x[n]$?

Reconstruction of an aperiodic signal

$x_p[n]$ is a periodic extension of $x[n] \Rightarrow$
$\Rightarrow x[n]$ can be reconstructed if there is no aliasing in the time domain.



$x[n]$ must be time-limited to less than the period N of $x_p[n]$

Reconstruction of an aperiodic signal

Given a finite-duration sequence $x[n]$ which is non-zero at the interval $0 \leq n \leq L-1$, then, for any $N \geq L$

$$x[n] = x_p[n] \qquad 0 \leq n \leq N-1$$

so that $x[n]$ can be recovered from its N-periodic extension $x_p[n]$ without ambiguity.

Sampling in frequency domain

The spectrum of an aperiodic discrete-time signal with finite duration L can be exactly recovered from its samples at frequencies $\omega_k = 2\pi k/N$, if $N \geq L$.

Frequency domain sampling

Let's consider the spectrum of a discrete-time aperiodic signal $x[n]$! Its spectrum is

- continuous (why?)
- periodic (why?)

We want to represent the continuous spectrum with its samples in the frequency domain:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \rightarrow X[k\delta\omega]$$

Fundamental questions (analogous to time-domain sampling):

- How many samples do we need?
- **How to reconstruct the continuous spectrum?**

Reconstruction procedure

$$
\begin{aligned}
X(\omega) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=0}^{N-1} x_p[n] e^{-j\omega n} = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N} k\right) e^{j2\pi kn/N} \right] e^{-j\omega n} \\
&= \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N} k\right) \left[\frac{1}{N} \sum_{n=0}^{N-1} e^{-j(\omega - 2\pi k/N)n} \right]
\end{aligned}
$$

Reconstruction procedure

DTFT

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=0}^{N-1} x_p[n]e^{-j\omega n} = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right)e^{j2\pi kn/N}\right]e^{-j\omega n}$$

$$= \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right)\left[\frac{1}{N}\sum_{n=0}^{N-1} e^{-j(\omega-2\pi k/N)n}\right]$$

Reconstruction procedure

DTFT

$x[n]$ is zero outside the interval. Inside it is equal to $x_p[n]$

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Reconstruction procedure

DTFT

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Reconstruction formula for the periodic extension

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=0}^{N-1} x_p[n]e^{-j\omega n} = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right)e^{j2\pi kn/N}\right]e^{-j\omega n}$$

$$= \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right)\left[\frac{1}{N}\sum_{n=0}^{N-1} e^{-j(\omega-2\pi k/N)n}\right]$$

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$$= \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) \left[\frac{1}{N} \sum_{n=0}^{N-1} e^{-j(\omega-2\pi k/N)n} \right]$$

Let's organize it to the form of the TD interpolation formula!

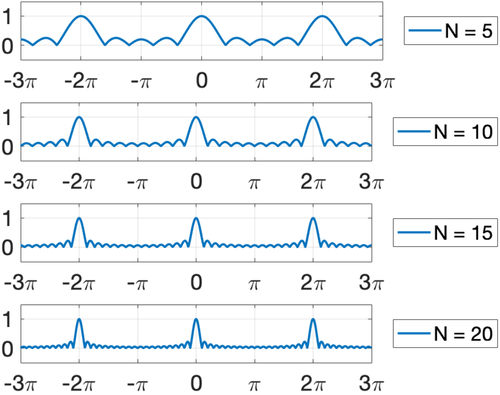
Reconstruction procedure

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=0}^{N-1} x_p[n]e^{-j\omega n} = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right)e^{j2\pi kn/N}\right]e^{-j\omega n}$$

$$= \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right)\left[\frac{1}{N}\sum_{n=0}^{N-1} e^{-j(\omega-2\pi k/N)n}\right] = \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right)P\left(\omega-\frac{2\pi}{n}k\right), \text{ where}$$

$$P(\omega) \equiv \frac{1}{N}\sum_{n=0}^{N-1} e^{-j\omega n} = \frac{1}{N}\frac{1-e^{-j\omega N}}{1-e^{-j\omega}} = \frac{1}{N}\frac{sin(\omega N/2)}{sin(\omega/2)}e^{-j\omega(N-1)/2}$$

Interpolation function



How does this compare with the ideal time-domain interpolation function $g(t) = \frac{\sin(\pi/T)t}{(\pi/T)t}$?

$$|P(\omega)| = |\frac{1}{N}\frac{\sin(\omega N/2)}{\sin(\omega/2)}e^{-j\omega(N-1)/2}|$$

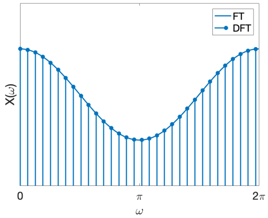
Discrete Fourier Transform

We have shown that, given a discrete time signal $x[n]$, it is possible to reconstruct its spectrum $X(\omega)$ from its N equidistant samples in case $x[n]$ is of finite length with $n = 0, 1, ..., L-1$ and $N \geq L$.

Definition

The Discrete Fourier Transform (DFT) of a sequence $x[n]$ is

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi \frac{kn}{N}}, \text{ for } 0 \leq k \leq N-1$$



Inverse Discrete Fourier Transform

We have also shown that the discrete time signal $x[n]$ can be reconstructed from N equidistant samples of its spectrum $X(\omega)$ in case $x[n]$ has finite length with $n = 0, 1, ..., L-1$ and $N \geq L$

Definition

The DFT values (i.e. $X[k]$, $0 \leq k \leq N-1$) uniquely define the discrete time signal $x[n]$, $n = 0, 1, ...L-1$, $L \leq N$ through the Inverse Discrete Fourier Transform (IDFT):

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi \frac{kn}{N}}, \text{ for } 0 \leq n \leq N-1$$

DFT as a linear transform

After defining $W_N = e^{-j2\pi/N}$, the DFT and IDFT can be expressed as:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad k = 0, 1, \ldots N-1$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \qquad n = 0, 1, \ldots N-1$$

Let's introduce:

$$\mathbf{x}_N = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}, \quad \mathbf{W}_N = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \ldots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \ldots & W_N^{(N-1)} \\ W_N^0 & W_N^2 & W_N^4 & \ldots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_N^0 & W_N^{(N-1)} & W_N^{(2N-1)} & \ldots & W_N^{(N-1)(N-1)} \end{bmatrix}, \quad \mathbf{X}_N = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix}$$

DFT as a linear transform

Then we can write the DFT and IDFT using the matrix notation:

(1) DFT: $\mathbf{X}_N = \mathbf{W}_N \mathbf{x}_N$

(2) IDFT: $\mathbf{x}_N = \frac{1}{N}\mathbf{W}_N^H \mathbf{X}_N$, where $()^H$ denotes the compex conjugate

Expanding (1):

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \dots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \dots & W_N^{(N-1)} \\ W_N^0 & W_N^2 & W_N^4 & \dots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_N^0 & W_N^{(N-1)} & W_N^{(2N-1)} & \dots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}$$

DFT as a linear transformation

Observations:

- The elements on the first row and columns are 1: $\mathbf{W}_N^0 = 1$.
- $\mathbf{W}_N$ is a symmetric matrix

Assuming that $\mathbf{W}_N$ is invertible, then

$$\begin{aligned}
\mathbf{X}_N &= \mathbf{W}_N \mathbf{x}_N \\
\color{red}{\mathbf{W}_N^{-1}}\mathbf{X}_N &= \color{red}{\mathbf{W}_N^{-1}}\mathbf{W}_N \mathbf{x}_N \\
\mathbf{W}_N^{-1}\mathbf{X}_N &= \mathbf{x}_N
\end{aligned}$$

We have established that $\mathbf{x}_N = \frac{1}{N}\mathbf{W}_N^H \mathbf{X}_N$. Therefore, $\mathbf{W}_N^{-1}$ exists and

$$\mathbf{W}_N^{-1} = \frac{1}{N}\mathbf{W}_N^H \Rightarrow \mathbf{W}_N \mathbf{W}_N^H = N\mathbf{I}_N$$

DFT as a linear transformation

Let's define the normalized DFT matrix as $N^{-\frac{1}{2}}\mathbf{W}_N$. Then

$$(N^{-\frac{1}{2}}\mathbf{W}_N)(N^{-\frac{1}{2}}\mathbf{W}_N^H) = \mathbf{I}_N$$

,i.e. the normalized DFT matrix is unitary. It follows that the columns of the matrix are orthonormal.

DFT as a projection on an orthonormal basis

The columns of the normalized DFT matrix form an orthonormal basis in a complex N-dimensional vector space. Therefore, the DFT coefficient, i.e the values $X[k]$ can be viewed as the coordinates of $x[n]$ in this basis (up to the constant factor $N^{-\frac{1}{2}}$).

Example 1

Let's compute the four-point DFT of a sequence: $x[n] = [0, 1, 2, 3]$

Solution:

1. Determine W_4!

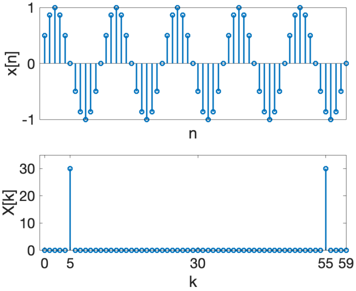
$$W_4 = \begin{bmatrix} W_4^{0\cdot 0} & W_4^{0\cdot 1} & W_4^{0\cdot 2} & W_4^{0\cdot 3} \\ W_4^{1\cdot 0} & W_4^{1\cdot 1} & W_4^{1\cdot 2} & W_4^{1\cdot 3} \\ W_4^{2\cdot 0} & W_4^{2\cdot 1} & W_4^{2\cdot 2} & W_4^{2\cdot 3} \\ W_4^{3\cdot 0} & W_4^{3\cdot 1} & W_4^{3\cdot 2} & W_4^{3\cdot 3} \end{bmatrix} = \begin{bmatrix} W_4^{0} & W_4^{0} & W_4^{0} & W_4^{0} \\ W_4^{0} & W_4^{1} & W_4^{2} & W_4^{3} \\ W_4^{0} & W_4^{2} & W_4^{0} & W_4^{2} \\ W_4^{0} & W_4^{3} & W_4^{2} & W_4^{1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & j \end{bmatrix}$$

2. $X_4 = W_4 x_4 = \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$

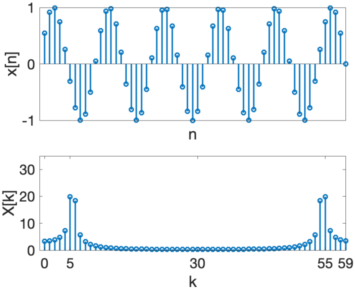
Example 2

$$x[n] = sin(\theta n)\ ,\ 0 \leq n \leq N-1\ ,\ N = 60$$

Case 1:
θ is an integer multiple of $2\pi/N$
e.g. $\theta = 2\pi \cdot 5/N$



Case 2:
θ is not an integer multiple of $2\pi/N$
e.g. $\theta = 2\pi \cdot 5.5/N$



Example 2 - Explanation

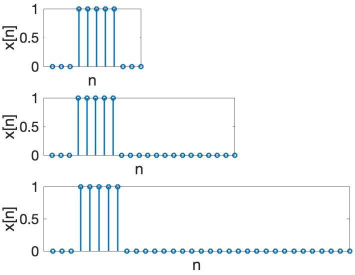
In DFT synthesis, the samples of $x[n]$ are a linear combination of orthogonal basis vectors:

$$x[n] = \sum_{k=0}^{N-1} X[k] e^{j2\pi \frac{kn}{N}}, \text{ for } 0 \leq n \leq N-1$$

- If the signal frequency θ matches one of the basis functions' frequency, then $x[n]$ can be written by scaling that particular basis function (and its complex conjugate). All other coefficients are zero.
- If the continuous spectrum was sampled in such a way that none of the basis functions match the signal frequency, then all basis functions are needed in the linear combination.

Alternative explanation: The periodic extension of the second sequence is not a samples sine wave!

DFT and zero-padding



Zero-padding

- Does not provide extra information: spectrum can be recovered already from only L samples of the spectrum
- Does improve visualization
- IDFT will gives back the zero-padded sequence

Properties of the DFT

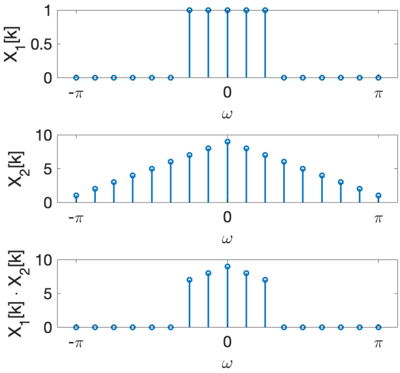
TABLE 7.2 Properties of the DFT

Property	Time Domain	Frequency Domain
Notation	$x(n),\ y(n)$	$X(k),\ Y(k)$
Periodicity	$x(n) = x(n+N)$	$X(k) = X(k+N)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(k) + a_2X_2(k)$
Time reversal	$x(N-n)$	$X(N-k)$
Circular time shift	$x((n-l))_N$	$X(k)e^{-j2\pi kl/N}$
Circular frequency shift	$x(n)e^{j2\pi ln/N}$	$X((k-l))_N$
Complex conjugate	$x^*(n)$	$X^*(N-k)$
Circular convolution	$x_1(n)\ Ⓝ\ x_2(n)$	$X_1(k)X_2(k)$
Circular correlation	$x(n)\ Ⓝ\ y^*(-n)$	$X(k)Y^*(k)$
Multiplication of two sequences	$x_1(n)x_2(n)$	$\frac{1}{N}X_1(k)\ Ⓝ\ X_2(k)$
Parseval's theorem	$\sum_{n=0}^{N-1} x(n)y^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$

- See also table 7.1 - symmetry properties
- Pratice: Brightspace quiz

Multiplication of DFTs

Given two finite duration sequences $x_1[n]$ and $x_2[n]$ and their N-point DFTs. The product of these DFTs, $X_3[k] = X_1[k] \cdot X_2[k]$ is the DFT of a third sequence $x_3[k]$.



Can we write $x_3[n]$
in terms of $x_2[n]$ and $x_3[n]$?

Multiplication of DFTs

$$x_3[m] = \frac{1}{N}\sum_{k=0}^{N-1} X_3[k] e^{j2\pi km/N} = \frac{1}{N}\sum_{k=0}^{N-1} X_1[k] X_2[k] e^{j2\pi km/N} =$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x_1[n] e^{-j2\pi kn/N}\right]\left[\sum_{l=0}^{N-1} x_2[l] e^{-j2\pi kl/N}\right] e^{j2\pi km/N}$$

$$= = \frac{1}{N}\sum_{n=0}^{N-1} x_1[n] \sum_{l=0}^{N-1} x_2[l] \left[\sum_{m=0}^{N-1} e^{j2\pi k(m-n-l)/N}\right] =$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x_1[n] \sum_{l=0}^{N-1} x_2[l] \sum_{k=0}^{N-1} a^k, \text{ with } a = e^{j2\pi(m-n-l)/N}$$

Multiplication of DFTs

$$x_3[m] = \frac{1}{N}\sum_{k=0}^{N-1} X_3[k]e^{j2\pi km/N} = \frac{1}{N}\sum_{k=0}^{N-1} X_1[k]X_2[k]e^{j2\pi km/N} =$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x_1[n]e^{-j2\pi kn/N}\right]\left[\sum_{l=0}^{N-1} x_2[l]e^{-j2\pi kl/N}\right]e^{j2\pi km/N}$$

$$= = \frac{1}{N}\sum_{n=0}^{N-1} x_1[n]\sum_{l=0}^{N-1} x_2[l]\left[\sum_{m=0}^{N-1} e^{j2\pi k(m-n-l)/N}\right] =$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x_1[n]\sum_{l=0}^{N-1} x_2[l]\sum_{k=0}^{N-1} a^k, \text{ with } a = e^{j2\pi(m-n-l)/N}$$

Multiplication of DFTs

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N & a = 1 \\ \frac{1-a^N}{1-a} & a \neq 1 \end{cases}$$

$$a = e^{j2\pi(m-n-l)/N}$$

(i) $a = e^{j2\pi(m-n-l)/N} = 1$ if $(m-n-l)/N$ is an integer

(ii) $a^N = \left(e^{j2\pi(m-n-l)/N}\right)^N = e^{j2\pi(m-n-l)} = 1$ if $a \neq 0$

Multiplication of DFTs

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N & a = 1 \\ \frac{1-a^N}{1-a} & a \neq 1 \end{cases} = \begin{cases} N & l = m - n + pN \text{ with p an integer} \\ 0 & \textit{otherwise} \end{cases}$$

$$a = e^{j2\pi(m-n-l)/N}$$

$$(i) \quad a = e^{j2\pi(m-n-l)/N} = 1 \text{ if } (m-n-l)/N \text{ is an integer}$$

$$(ii) \; a^N = \left(e^{j2\pi(m-n-l)/N}\right)^N = e^{j2\pi(m-n-l)} = 1 \text{ if } a \neq 0$$

Multiplication of DFTs

$$
\begin{aligned}
x_3[m] &= \frac{1}{N}\sum_{k=0}^{N-1} X_3[k]e^{j2\pi km/N} = \frac{1}{N}\sum_{k=0}^{N-1} X_1[k]X_2[k]e^{j2\pi km/N} = \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x_1[n]e^{-j2\pi kn/N}\right]\left[\sum_{l=0}^{N-1} x_2[l]e^{-j2\pi kl/N}\right]e^{j2\pi km/N} = \\
&= \frac{1}{N}\sum_{n=0}^{N-1} x_1[n]\sum_{l=0}^{N-1} x_2[l]\left[\sum_{m=0}^{N-1} e^{j2\pi k(m-n-l)/N}\right] = \\
&= \frac{1}{N}\sum_{n=0}^{N-1} x_1[n]\sum_{l=0}^{N-1} x_2[l]{\color{red}\sum_{k=0}^{N-1} a^k} = \begin{cases} \sum_{n=0}^{N-1} x_1[n]\sum_{l=0}^{N-1} x_2[l], & l = m-n+pN \\ 0, & otherwise \end{cases}
\end{aligned}
$$

Multiplication of DFTs

$$x_3[m] = \frac{1}{N}\sum_{k=0}^{N-1} X_3[k]e^{j2\pi km/N} = \frac{1}{N}\sum_{k=0}^{N-1} X_1[k]X_2[k]e^{j2\pi km/N} =$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x_1[n]e^{-j2\pi kn/N}\right]\left[\sum_{l=0}^{N-1} x_2[l]e^{-j2\pi kl/N}\right]e^{j2\pi km/N} =$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x_1[n]\sum_{l=0}^{N-1} x_2[l]\left[\sum_{m=0}^{N-1} e^{j2\pi k(m-n-l)/N}\right] =$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x_1[n]\sum_{l=0}^{N-1} x_2[l]\sum_{k=0}^{N-1} a^k = \begin{cases} \sum_{n=0}^{N-1} x_1[n]\sum_{l=0}^{N-1} x_2[l], & l = m - n + pN \\ 0, & otherwise \end{cases}$$

For a given m and n, $l = m + n + pN$ can only be true for one specific p, because $l \leq N$

Multiplication of DFTs

$$x_3[m] = \frac{1}{N}\sum_{k=0}^{N-1} X_3[k]e^{j2\pi km/N} = \frac{1}{N}\sum_{k=0}^{N-1} X_1[k]X_2[k]e^{j2\pi km/N} =$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x_1[n]e^{-j2\pi kn/N}\right]\left[\sum_{l=0}^{N-1} x_2[l]e^{-j2\pi kl/N}\right]e^{j2\pi km/N} =$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x_1[n]\sum_{l=0}^{N-1} x_2[l]\left[\sum_{m=0}^{N-1} e^{j2\pi k(m-n-l)/N}\right] =$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x_1[n]\sum_{l=0}^{N-1} x_2[l]\sum_{k=0}^{N-1} a^k = \sum_{n=0}^{N-1} x_1[n]x_2[m-n+pN]$$

For p=0 this would be a convolution sum. However, the index of x_2 is always shifted back to the interval between 0 and N-1. This can be expressed using the modulo operator.

Multiplication of DFTs

$$x_3[m] = \frac{1}{N}\sum_{k=0}^{N-1} X_3[k]e^{j2\pi km/N} = \frac{1}{N}\sum_{k=0}^{N-1} X_1[k]X_2[k]e^{j2\pi km/N} =$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x_1[n]e^{-j2\pi kn/N}\right]\left[\sum_{l=0}^{N-1} x_2[l]e^{-j2\pi kl/N}\right]e^{j2\pi km/N} =$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x_1[n]\sum_{l=0}^{N-1} x_2[l]\left[\sum_{m=0}^{N-1} e^{j2\pi k(m-n-l)/N}\right] =$$

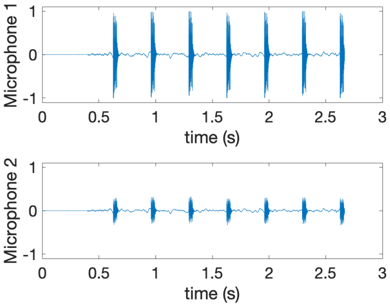
$$= \frac{1}{N}\sum_{n=0}^{N-1} x_1[n]\sum_{l=0}^{N-1} x_2[l]\sum_{k=0}^{N-1} a^k = \sum_{n=0}^{N-1} x_1[n]x_2[m-n+pN]$$

$$\equiv \sum_{n=0}^{N-1} x_1[n]x_2[m-n]_N$$

Multiplication of the DFT of 2 sequences is the so-called circular convolution in time domain.

Multiplication of DFTs - Example

Channel estimation (EE2T11):

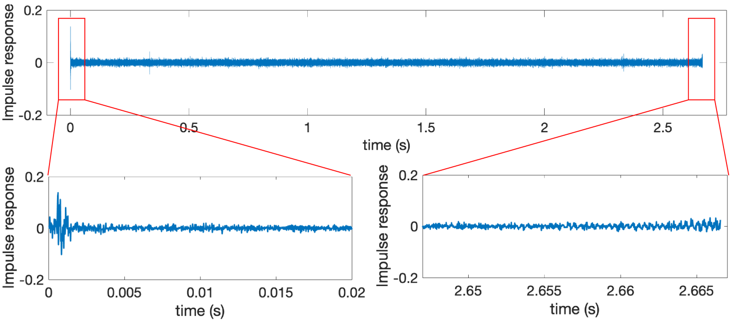


Deconvolution in the frequency domain:

1. Compute $X(k)$ and $Y(k)$, the DFTs of x and y (microphone close and far from the loudspeaker, resp.)
2. Compute $H(k) = \frac{Y(k)}{X(k)}$
3. Compute $h(k)$ from $H(k)$ using the IDFT

DFT - Example 4

Channel estimation result:



As expected, the impulse response tends to zero after the initial peak. However, an increase in values is observed in the end of the response. These values belong to the negative time of the impulse response. They appear in the end due to the circular convolution proptery of DFT.