Signal Processing EE2S31

Digital Signal Processing - Lecture 2: Non-ideal sampling and reconstruction

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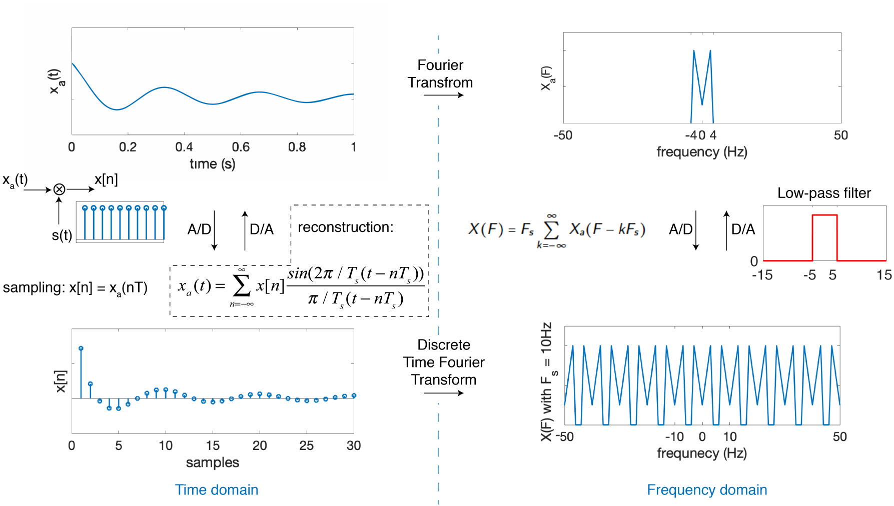
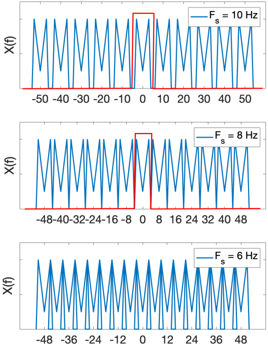
Recap: Ideal sampling and reconstruction



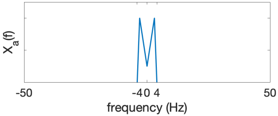
What do we see in this video?

Click here!

Recap: Spectrum of a sampled signal



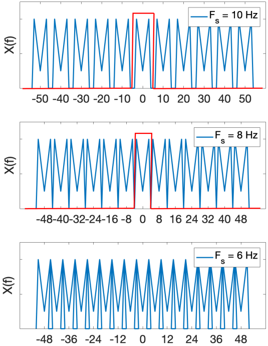
$$F_s \geq 2F_H$$



Sampling theorem

If the signal is bandlimited, it is possible to reconstruct the original signal from the samples, provided that the sampling rate is at least twice the highest frequency contained in the signal (i.e. the Nyquist rate).

Recap: Spectrum of a sampled signal



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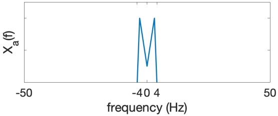

otherwise: Aliasing!

Sampling theorem

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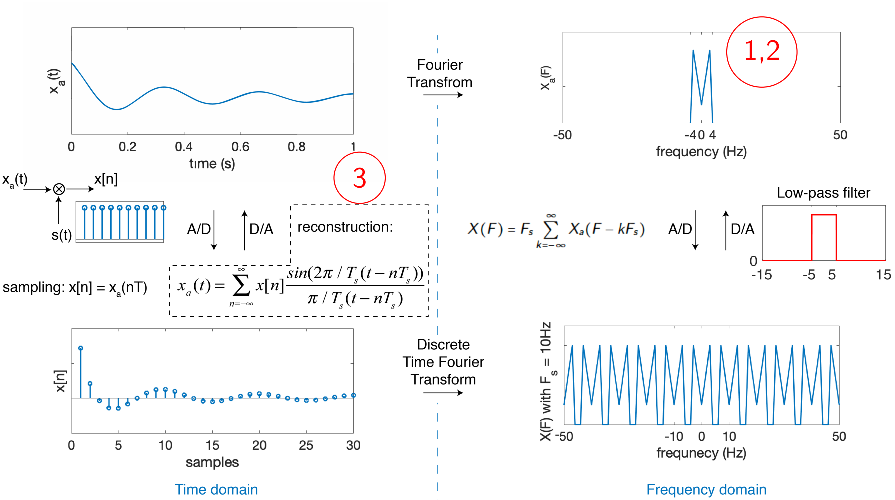
Non-ideal sampling and reconstruction



Goal of this lecture

- Part 1: Non-bandlimited signals
    - Explain the phenomenon of aliasing
    - Anti-aliasing filters
- Part 2: Bandpass signals
    - How to sample bandpass signals?
    - How to reconstruct bandpass signals?
- Part 3: Reconstruction in practice - linear interpolation

Part 1

Aliasing

- Occurs when sampling with Sampling with $F_s < 2F_H$!

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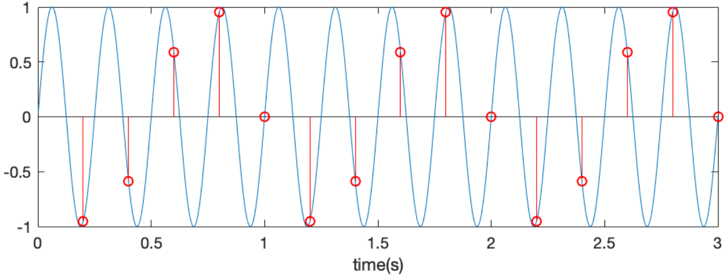
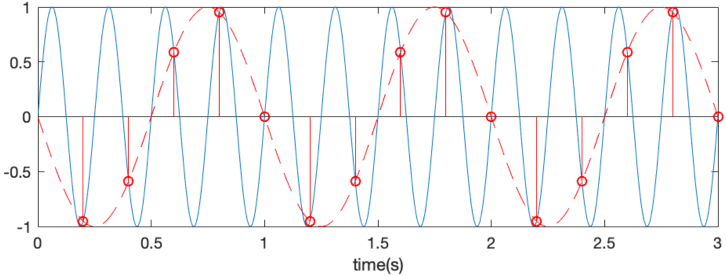
- Effect of aliasing: example 1



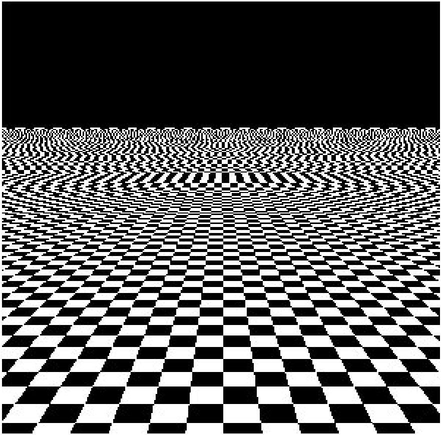
Aliasing

- Occurs when sampling with Sampling with $F_s < 2F_H$!
- Effect of aliasing: example 1



Aliasing

- Occurs when sampling with Sampling with $F_s < 2F_H$!
- Effect of aliasing: example 2



Aliasing

- Occurs when sampling with Sampling with $F_s < 2F_H$!
- Effect of aliasing: example 3

Landline phone audio signal sampled at 8kHz!

Original speech

Sampled speech

Sampling after prefilter

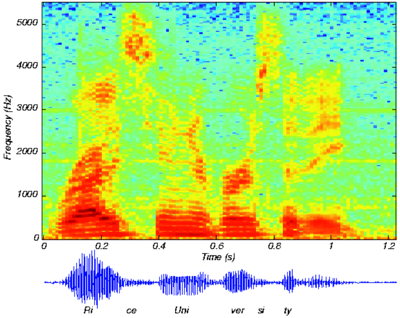
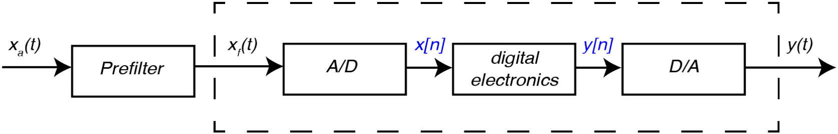
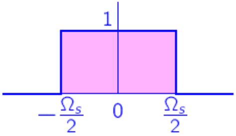


image source

Sampling of non-bandlimited signals

To avoid aliasing when sampling at a rate Ω_s, a "prefilter" or "antialiasing filter" $H_{aa}(\Omega)$ with a cut-off at $\Omega_s/2$ should be used prior to sampling:





$$H_{aa}(\Omega) = \begin{cases} 1, & \text{if } |\Omega| \le \Omega_s/2 \\ 0, & \text{otherwise} \end{cases}$$

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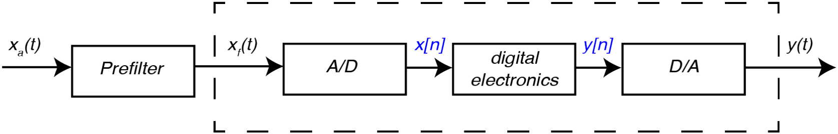
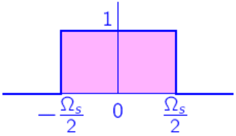




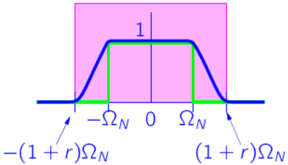
$$H_{aa}(\Omega) = \begin{cases} 1, & \text{if } |\Omega| \leq \Omega_s/2 \\ 0, & \text{otherwise} \end{cases}$$

However, a very sharp filter in the analog domain is difficult to implement.

Antialiasing in practice

Solution 1:

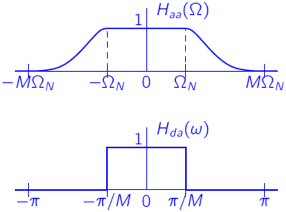
- Choose a desired Ω_N (e.g. 20kHz, then the Nyquist rate would be $2 \cdot \Omega_N = 40$ kHz)
- Take a somewhat larger sampling rate $\Omega_s = 2 \cdot (1 + r)\Omega_N$ with $0 < r < 1$ (e.g. 44,1kHz $\rightarrow$ CD!)
- Make use of a non-ideal lowpass filter with a transition band between Ω_N and $(1 + r)\Omega_N = \Omega_s/2$
- filter further in digital domain if needed



Antialiasing in practice

Solution 2:

- Use a cheap antilaliasing filter with a broad transition band
- oversample $x_a(t)$ with a factor $2M$: $\Omega_s = 2M\Omega_N$
- digitally filter unwanted frequencies, where sharp filters are cheaper to implement
- downsample with a factor M (see multirate systems!)

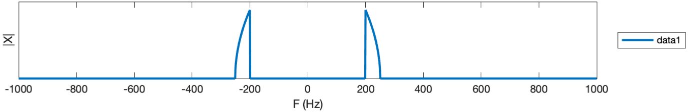


Part 2

Sampling of bandpass signals

Bandpass signal

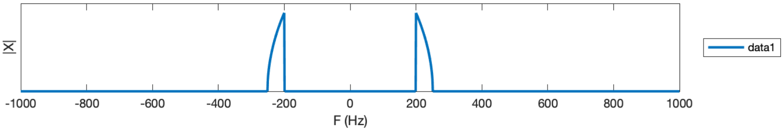
A bandpass signal with bandwidth B and center frequency F_c is a signal with nonzero spectral content at frequencies F defined by $0 < F_L < |F| < F_H$, where $F_c = \frac{F_L+F_H}{2}$ and $B = F_H - F_L$.



$$F_L = 200Hz,\ F_H = 250Hz,\ B = 50Hz,\ F_c = 225Hz$$

According to the sampling theory, we should sample with $F_s = 500Hz$.

Nyquist sampling of a bandpass signal



$$\Downarrow X(F) = F_s \sum_{k=-\infty}^{\infty} X_a(F - kF_s) \text{ with } F_s = 500Hz$$

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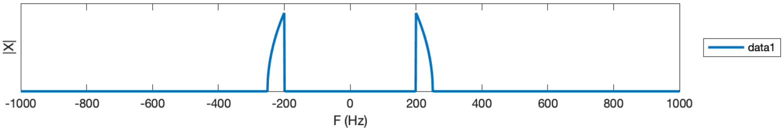

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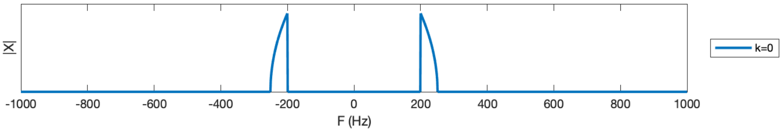

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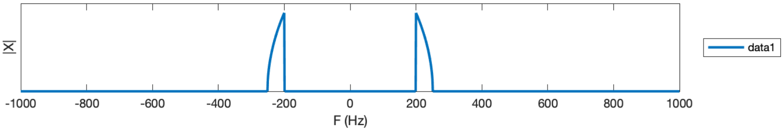

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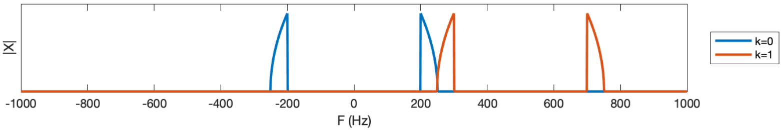

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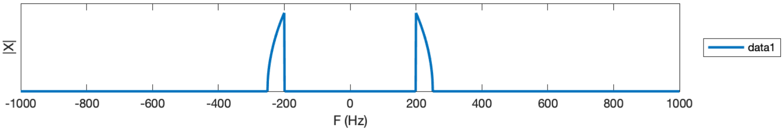

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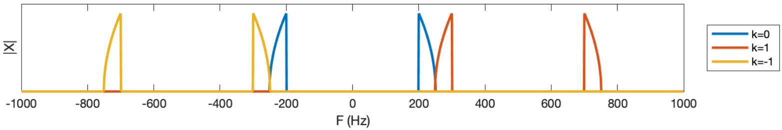

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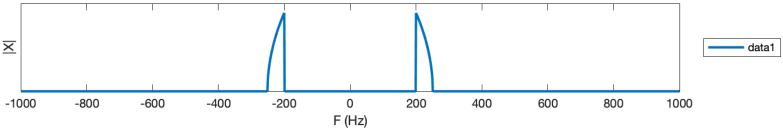

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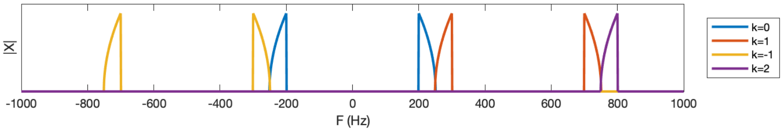

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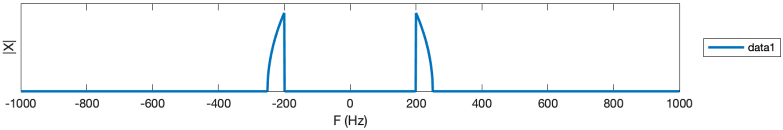

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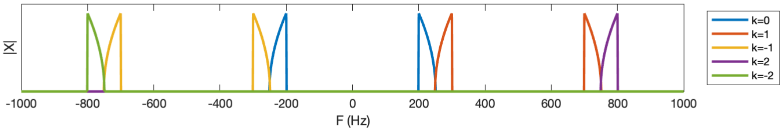
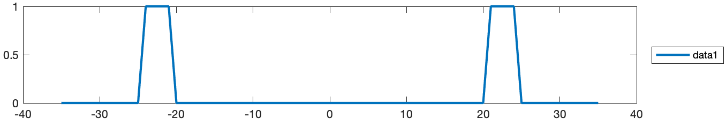


Example: spinning wheel

Let's assume that the car drives on the highway with 100-120km/h and has a 16 inch wheel (20-25Hz)!

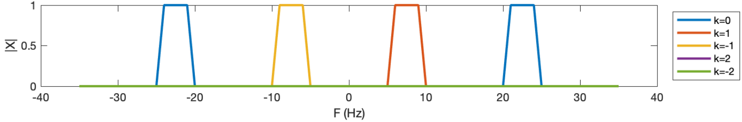


Exercise:

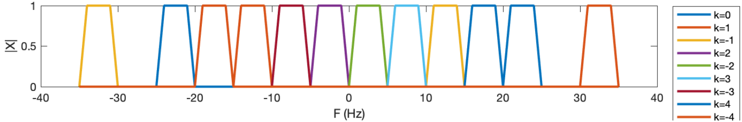
- Let's assume that we sample the signal with $F_s = 8Hz$, $F_s = 10Hz$ and $F_s = 30Hz$.
  Sketch the resulting spectrum!
- Answer the following questions (for each F_s):
  1. Is the Nyquist rate respected?
  2. Is there aliasing?
  3. Can we reconstruct the original signal?
  4. Can you suggest other sampling rates that will enable reconstruction?

Solution

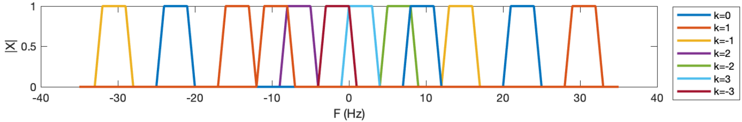
$F_s = 30Hz$:



$F_s = 10Hz$:



$F_s = 8Hz$:

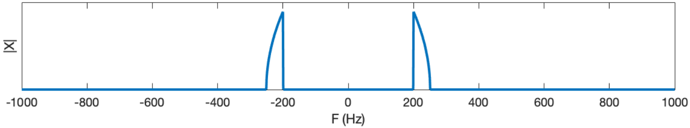


The Nyquist rate is not respected in any of the three cases. Aliasing occurs in each case, yet, it is possible to reconstruct, except for $F_s = 8Hz$ (=destructive aliasing)

Sampling of a bandpass signal: example 1

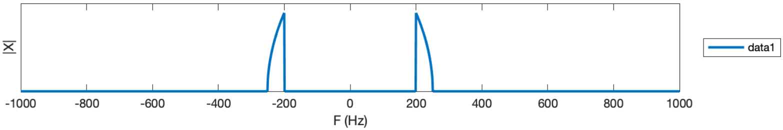
Integer band positioning

In case $F_H = mB$ sampling with $F_s = 2B$ is possible without aliasing.



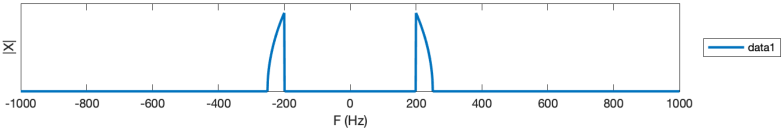
In the current example $F_H = 250$, $B = 50$ and $m = 5$.

Sampling of a bandpass signal: example 1



$$\Downarrow X(F) = F_s \sum_{k=-\infty}^{\infty} X_a(F - kF_s) \text{ with } F_s = 2B = 100Hz$$

Sampling of a bandpass signal: example 1



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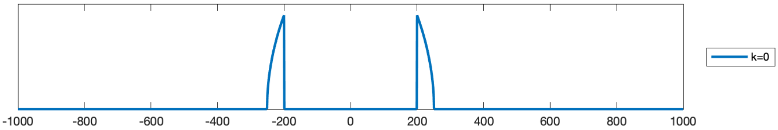

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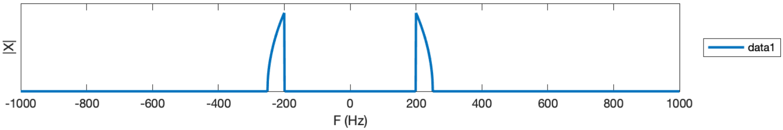

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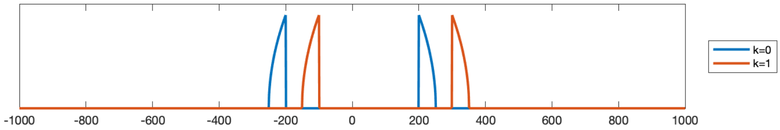

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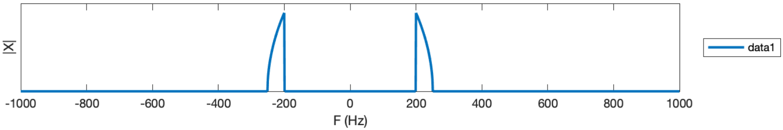

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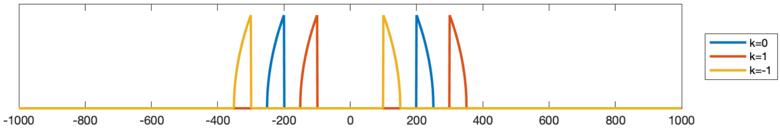

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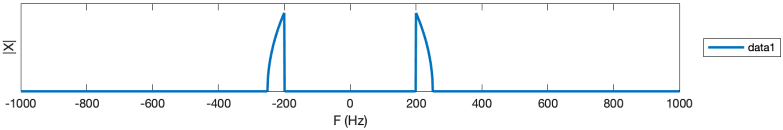

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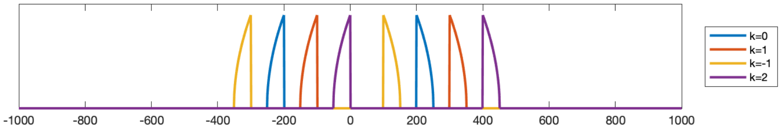

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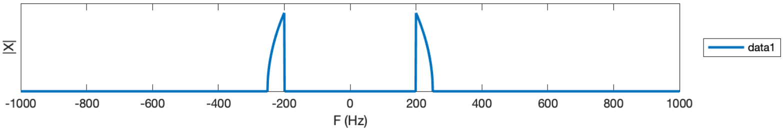

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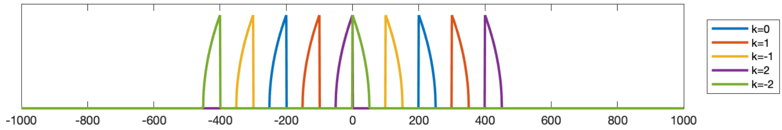

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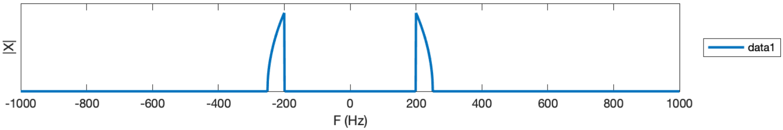

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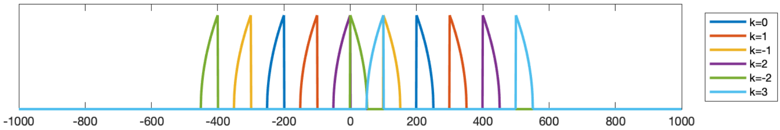

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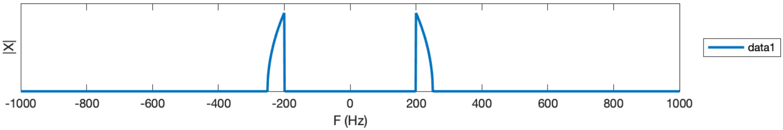

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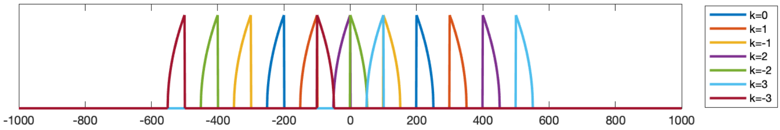

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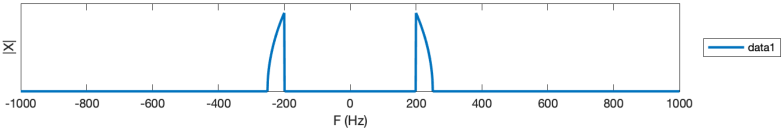

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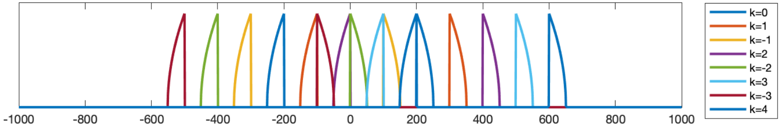

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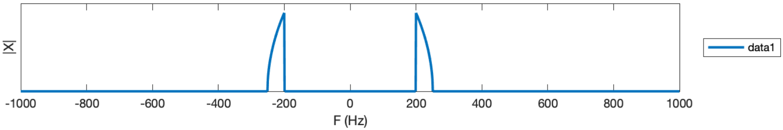

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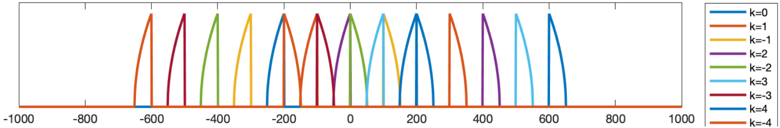
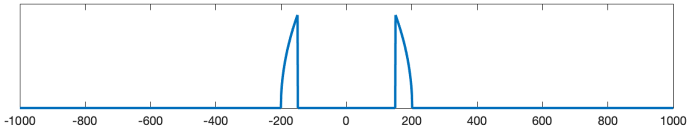
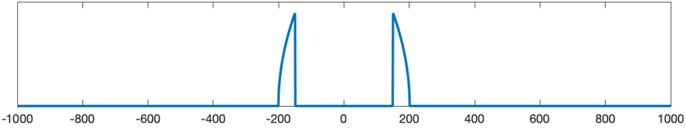


Sampling of a bandpass signal: example 2



In the second example $F_H = 200$, $B = 50$ and $m = 4$.

Sampling of a bandpass signal: example 2



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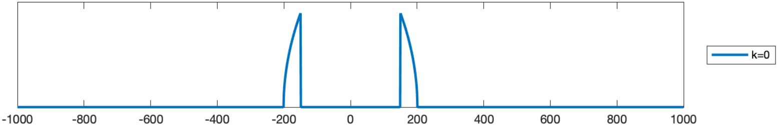
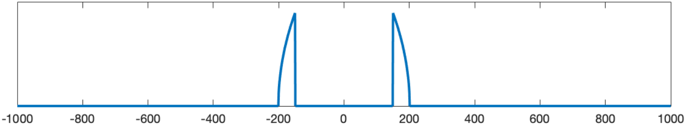
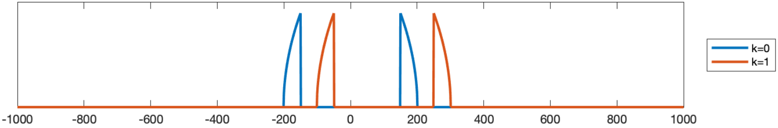


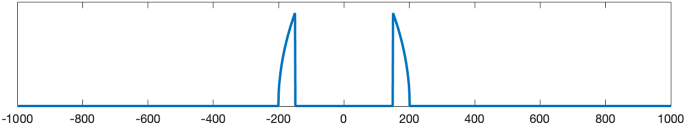
Sampling of a bandpass signal: example 2



$$\Downarrow X(F) = F_s \sum_{k=-\infty}^{\infty} X_a(F - kF_s) \text{ with } F_s = 2B = 100Hz$$



Sampling of a bandpass signal: example 2



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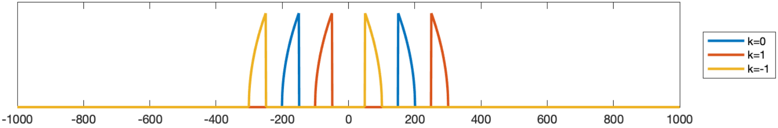
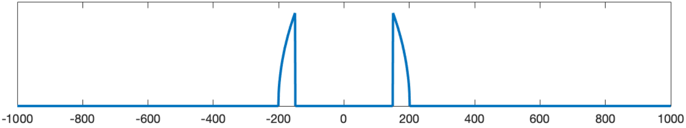


Sampling of a bandpass signal: example 2



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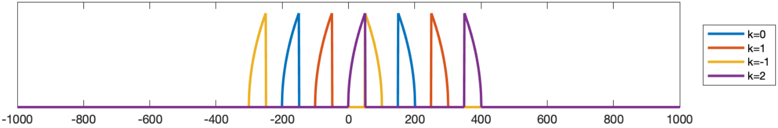

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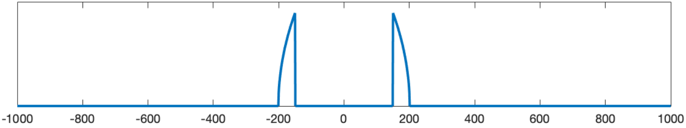

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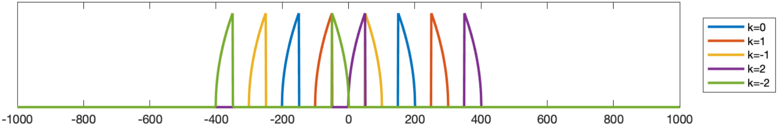
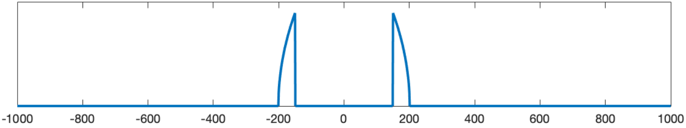


Sampling of a bandpass signal: example 2



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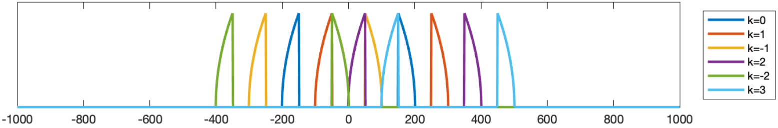
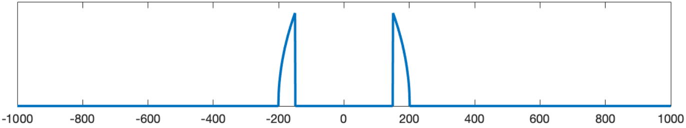


Sampling of a bandpass signal: example 2



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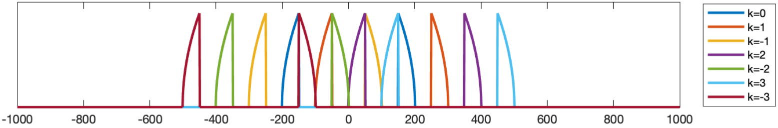
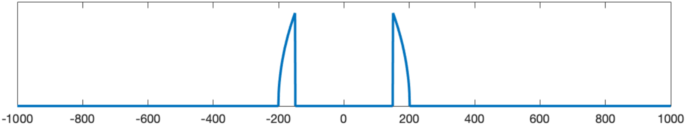


Sampling of a bandpass signal: example 2



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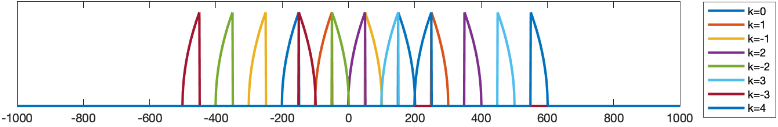
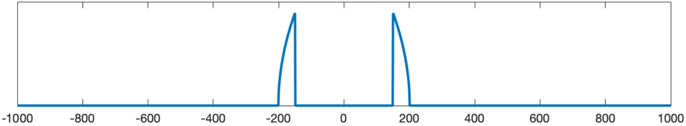


Sampling of a bandpass signal: example 2



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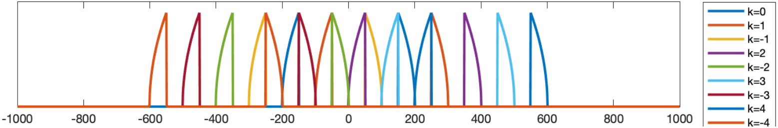
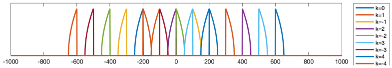
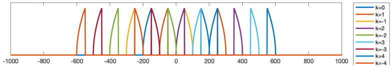
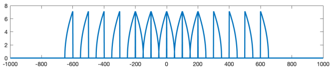


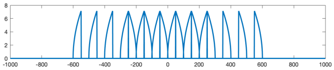
Reconstruction and downconversion





Reconstruction and downconversion





The specta of the sampled even and odd band positioned singals are both free from aliasing

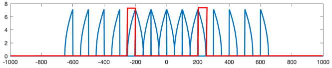
Reconstruction and downconversion

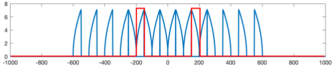
The original signal can be reconstructed using

$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a(nT)g(t - nT), \text{ with}$$

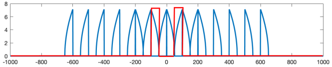
$$g(t) = \frac{\sin\pi Bt}{\pi Bt}\cos 2\pi F_c t$$

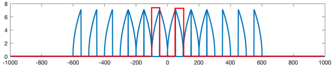
Note: $g(t)$ is equal to the interpolation function of bandlimited signals, modulated with the carrier frequency F_c





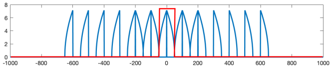
Reconstruction and downconversion

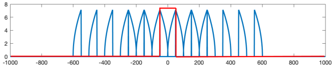




Downconversion: we may reconstruct a continuous bandpass signal centered at intermediate frequencies $F_{c'} = \pm(kB + B/2)$.

Reconstruction and downconversion





Downconversion: we may reconstruct a continuous bandpass signal centered at intermediate frequencies $F_{c'} = \pm(kB + B/2)$.

With $k = 0$ we obtain the equivalent *baseband* signal.

Note: the baseband spectra with even and odd band positions signal are 'inverted'.

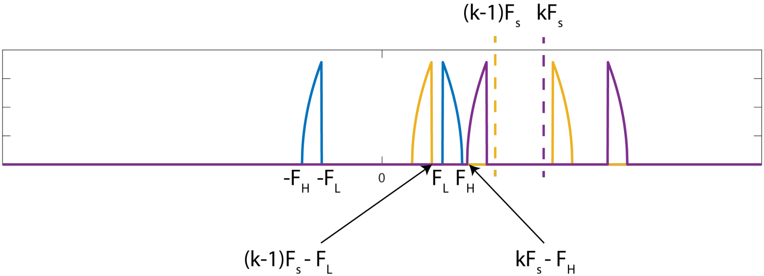
Example: radio receiver

FM radio uses VHF band 87.5 - 108 MHz. Each channel uses a 0.2MHz wide band. However, designing a filter with a tunable pass band is difficult.

- Analog radio:
  - Superheterodyne (Edwin Armstrong 1918):
  - recieved signal is shifted to a fixed intermediate frequency (IF) using a mixer with a tunable local oscillator (LO)
  - signal can be amplified at IF, demodulated and low-pass filtered to reconstruct the broadcast signal
- Digital radio:
  - choosing an appropriate sampling frequency will digitally downconvert the signal to baseband
  - reconstruction with a low-pass filter
  - Challenge: A/D conversion speed must be consistent with F_H
  - Other applications: radar, satellite communications, etc.
  - Further reading: Direct RF sampling

Arbitrary band positioning

- A signal has *arbitrary band positioning*, when there is no particular relationship between F_H and B (as opposed to integer band positioning)
- How to choose F_s in this case?



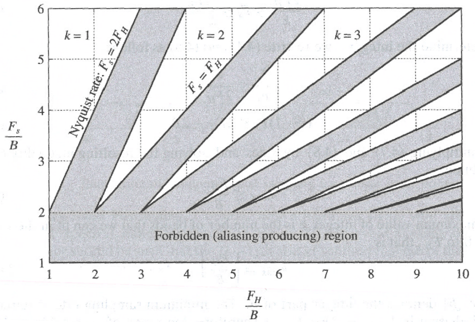
Conditions on F_s:

$$(k-1)F_s - F_L \leq F_L$$
$$kF_s - F_H \geq F_H$$

Arbitrary band positioning

Reorganizing the above conditions, we can arrive to the following expression:

$$\frac{2}{k}\frac{F_H}{B} \leq \frac{F_s}{B} \leq \frac{2}{k-1}\left(\frac{F_H}{B}-1\right),\ k_{max} = \left\lfloor \frac{F_H}{B} \right\rfloor$$



For our signal, we know F_H and B.

Then, we can choose an F_s/B along the vertical line corresponding to F_H/B

See problem 6.11

Part 3

Reconstruction in practice

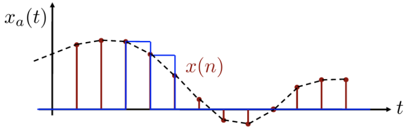
Ideal interpolation:

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] g_{ideal}(t - nT_s) = \sum_{n=-\infty}^{\infty} x[n] \frac{sin(\pi/T_s(t - nT_s))}{\pi/T_s(t - nT_s)}$$

However:

- sinc function is infinite and nondeterministic!
- in practice we sum from -L to L
- quality of reconstruction increases with L
- not practical in real-time applications
- instead: sample-and-hold (zero-order hold) or linear interpolation (first order hold)

Sample-and-hold interpolation



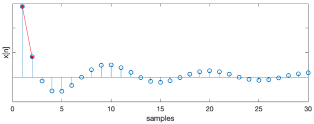
- time domain:

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] g_{SH}(t - nT_s) \ , \ g_{SH}(t) = \begin{cases} 1 & 0 \leq t \leq T_s \\ 0, & \text{otherwise} \end{cases}$$

- frequency domain:

$$G_{SH}(F) = \int_{-\infty}^{\infty} g_{SH}(t) e^{-j2\pi Ft} dt = T_s \frac{sin \pi F T_s}{\pi F T_s} e^{-j2\pi F(T_s/2)}$$

Linear interpolation

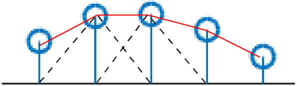


$$x_{lin}(t) = x[1] + \frac{x[2] - x[1]}{T_s}(t - T_s), \; T_s \leq t \leq 2T_s$$

In general:

$$x_{lin}(t) = x[n] + \frac{x[n+1] - x[n]}{T_s}(t - nT_s), \; nT_s \leq t \leq (n+1)T_s$$

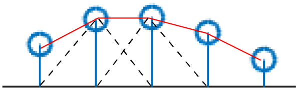
Linear interpolation



Reconstruction formula:

$$x_{lin}(t) = \sum_{n=-\infty}^{\infty} x[n] g_{lin}(t - nT_s)$$

Linear interpolation



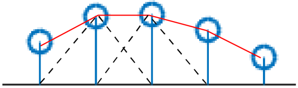
What should be g_{lin}?:

- $x(t)$ between $x[n]$ and $x[n-1]$ only depend in the value of these 2 samples
- "echo" of $x[n]$ does not extend beyond $x(t-T_s)$ or $x(t+T_s)$
- the sum of the "echos" of $x[n]$ and $x[n+1]$ are a linear function of t

Reconstruction formula:

$$x_{lin}(t) = \sum_{n=-\infty}^{\infty} x[n] g_{lin}(t - nT_s)$$

Linear interpolation



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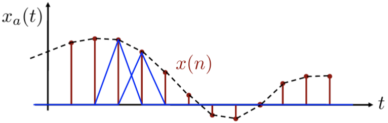
- the sum of the "echos" of $x[n]$ and $x[n+1]$ are a linear function of t

Reconstruction formula:

$$x_{lin}(t) = \sum_{n=-\infty}^{\infty} x[n] g_{lin}(t - nT_s), \text{ where } g_{lin}(t) = \begin{cases} 1 - \frac{|t|}{T_s}, & \text{if } |t| \leq T_s \\ 0, & \text{otherwise} \end{cases}$$

Linear interpolation



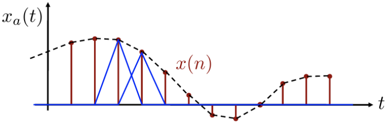
- time domain:

$$x_{lin}(t) = \sum_{n=-\infty}^{\infty} x[n] g_{lin}(t - nT_s), \text{ where } g_{lin}(t) = \begin{cases} 1 - \frac{|t|}{T_s}, & \text{if } |t| \leq T_s \\ 0, & \text{otherwise} \end{cases}$$

- frequency domain:

$$G_{lin}(f) = T_s [\frac{sin(\pi F T_s)}{\pi F T_s}]^2$$

Linear interpolation



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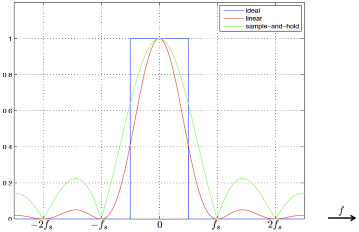
- frequency domain:

$$G_{lin}(f) = T_s [\frac{sin(\pi F T_s)}{\pi F T_s}]^2$$

- Problem 6.15: formulas are presented with a delay of T

Comparison of interpolation filters

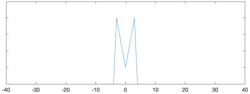


$$G_{ideal}(F) = \begin{cases} T_s, & |F| \leq F_s/2 \\ 0, & otherwise \end{cases}$$

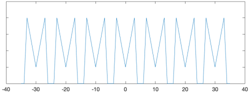
$$G_{SH}(F) = T_s \frac{sin(\pi F T_s)}{\pi F T_s} e^{-2j\pi F(T_s/2)}$$

$$G_{lin}(F) = T_s [\frac{sin(\pi F T)}{\pi F T_s}]^2$$

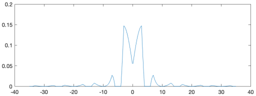
Distortion due to practical interpolation



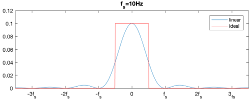
↓ sampling



↓ reconstruction



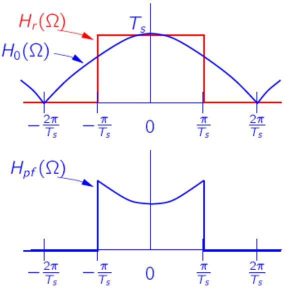
×



Compensation with a postfilter

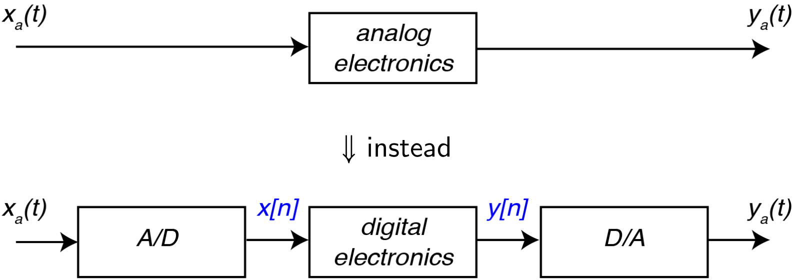
Let's denote the actual interpolation filter with H_0 and the ideal interpolation filter with H_r! The postfilter H_{pf} compensates for the difference:

$$H_{pf}(\Omega) = \frac{H_r(\Omega)}{H_0(\Omega)} = \begin{cases} T_s/H_0(\Omega), & \text{if } |\Omega| \leq \Omega_s/2 \\ 0, & \text{otherwise} \end{cases}$$



In practice, this filter is applied in the digital domain before the D/A

Summary



$x_a(t)$
analog electronics
$y_a(t)$
⇓ instead
$x_a(t)$
A/D
x[n]
digital electronics
y[n]
D/A
$y_a(t)$

Summary

Discrete time processing of continuous signals

Provided that the analog signal $x_a(t)$ is band-limited with bandwidth B and we sample with an $F_s \geq 2B$, then the discrete-time processing of $x[n] = x_a(nT_s)$ with a system $H(F)$ is equivalent to the analog processing of $x_a(t)$ with a system $H_a(F)$ in case $H_a(F) = H(F)$ for $|F| \leq F_s/2$ and $H_a(F) = 0$ otherwise.

Practical limitations and solutions:

- $x_a(t)$ is not (perfectly) band-limited → antialiasing filter
- A/D conversion: sampling is finite and not instantaneous!
- D/A converion: → interpolation and postfiltering