Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Signal Processing Systems

Resit exam EE2S31 SIGNAL PROCESSING
16 July 2025 13:30–16:30

Closed book; two sides of one A4 with handwritten notes permitted. No other tools except a basic pocket calculator permitted. Note the attached tables!

This exam consists of five questions (30 points). Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.

Question 1 (5 points)

An audio signal $x_a(t)$ is sampled at 40 kHz such that we obtain a sequence $x[n] = x_a(nT)$. We take samples during 2.5 seconds. The signal has to be filtered by an FIR filter $h[n]$ with 1024 coefficients.

(a) What is the highest admissible frequency in the signal $x_a(t)$?

(b) How many multiplications are needed to apply the filter to the data $x[n]$, using a direct implementation of the convolution? (Ignore side effects in this calculation.)

(c) We would like to use the FFT to reduce the number of operations. Following the overlap-add method, describe in detail how the convolution can be implemented using the FFT. Also give a block scheme.

(d) How many multiplications are needed now? [Assume that the FFT of order N has a complexity of $N \log_2(N)$.]

Solution

1p (a) 20 kHz.

1p (b) In total we have 100 kS (kilo-samples). For each sample we need 1024 multiplications ("flops"). In total: 102 Mflops.

1.5p (c) The signal is split into blocks of 1024 samples. Suppose that $x_i[n]$ contains the samples of the i-th block, followed by 1024 zeros to a length of 2048 samples (you could take a larger block as well.) We take the FFT of $x_i[n]$, call the result $X_i[k]$. It has 2048 samples.

The filter $h[n]$ is zero-padded to the same length of 2048 samples. Call its FFT $H[k]$ (also 2048 samples).

We apply pointwise multiplication: $Y_i[k] = H[k]X_i[k]$. Next, we compute its IFFT, resulting in $y_i[n]$ consisting of 2048 samples. This comprises 2 blocks and overlaps with $y_{i+1}[n]$.

Finally, we use linearity and add the overlapping parts of the $y_i[n]$.

Blockscheme: see slides on overlap-add.

1.5p (d) The FFT of $h[n]$ is done once ($N = 2048$).

The FFT of the $x_i[n]$: we have 98 blocks of 1024 samples, so we need 98 FFTs of size $N = 2048$.

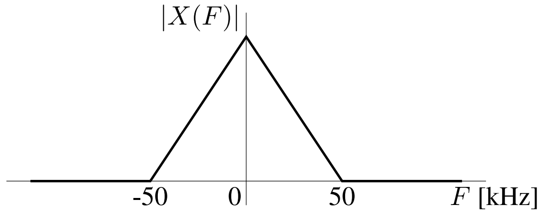
We need 98 times 2048 multiplications to form the $Y_i[k]$.

The IFFT of the $Y_i[k]$ requires 98 FFTs of size $N = 2048$.

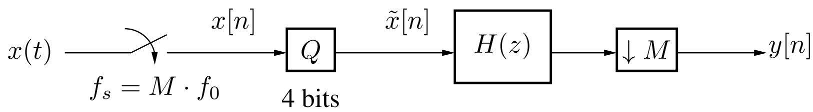
In total: $1 + 98 + 98 = 198$ times $N \log_2 N$ is $198 \times 2048 \times 11 = 4.5$ Mflops. Add to this the multiplications in the frequency domain: $98 \times N = 200$ kflops. Total: 4.7 Mflops.

Question 2 (9 points)

A signal $x(t)$ has an amplitude spectrum schematically shown as follows:



To sample $x(t)$, an oversampling AD converter operates at $M = 256$ times the desired sampling rate $f_0 = 100$ kHz, and quantizes samples at 4 bits. Digitally, the sample rate is reduced by a factor $M = 256$. See figure:



$H(z)$ is a lowpass filter given by $H(z) = 1 + z^{-1} + \cdots + z^{-(M-1)} = \dfrac{1 - z^{-M}}{1 - z^{-1}}$.

(a) Draw the amplitude spectrum $|H(e^{j\omega})|$. (Clearly label the frequency axis. For illustration purposes, you may consider a small M.)

In this plot, what is the location (value of ω) of the first zero of the transfer function?

(b) What is the role of this filter in the context of the oversampling ADC architecture?

If it was an ideal filter, what would be its specification?

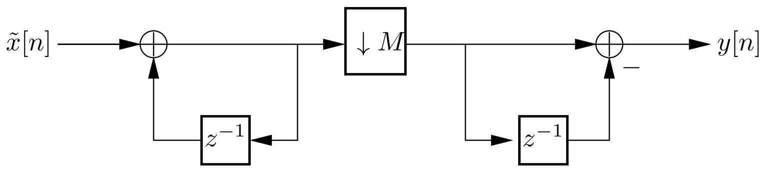
(c) What is the effect of this filter on the quantization noise?

(d) Draw amplitude spectra for the signals $x[n]$, $\tilde{x}[n]$, and $y[n]$. Label the frequency axis both in terms of ω [rad] and the corresponding original frequencies F [kHz].

(e) As usual, we model the effect of the quantization as additive white noise, with a uniform distribution. Its variance is $\sigma_e^2 = \Delta^2/12$ with $\Delta = R/2^{B+1}$ where R is the range of the quantizer and $B = 4$ is the number of bits. You may consider $R = 1$.

How many bits accuracy do you expect at the output? (Give a derivation.)

(f) An efficient implementation of the filter, in combination with the downsampler, is as follows:



Prove that this implementation indeed results in the desired response.

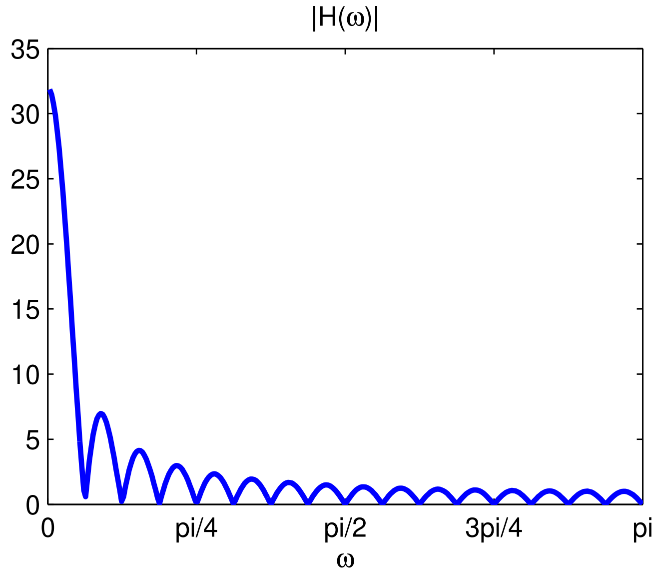
(*Hint:* recall the "noble identities".)

Solution

1p (a) Recognize the usual digital sinc function (Dirichlet function). We can write:

$$H(e^{j\omega}) = \frac{1 - e^{-jM\omega}}{1 - z^{-j\omega}} = \frac{\sin(M\omega/2)}{\sin(\omega/2)} e^{-j(M-1)\omega/2} .$$

$H(z)$ has M zeros, at $z = e^{j\frac{2\pi}{M}k}$ $(k = 0, \cdots, M-1)$, and a pole at $z = 1$ which will cancel the corresponding zero. The first zero is at $\omega = 2\pi/M$, which corresponds to 100 kHz.



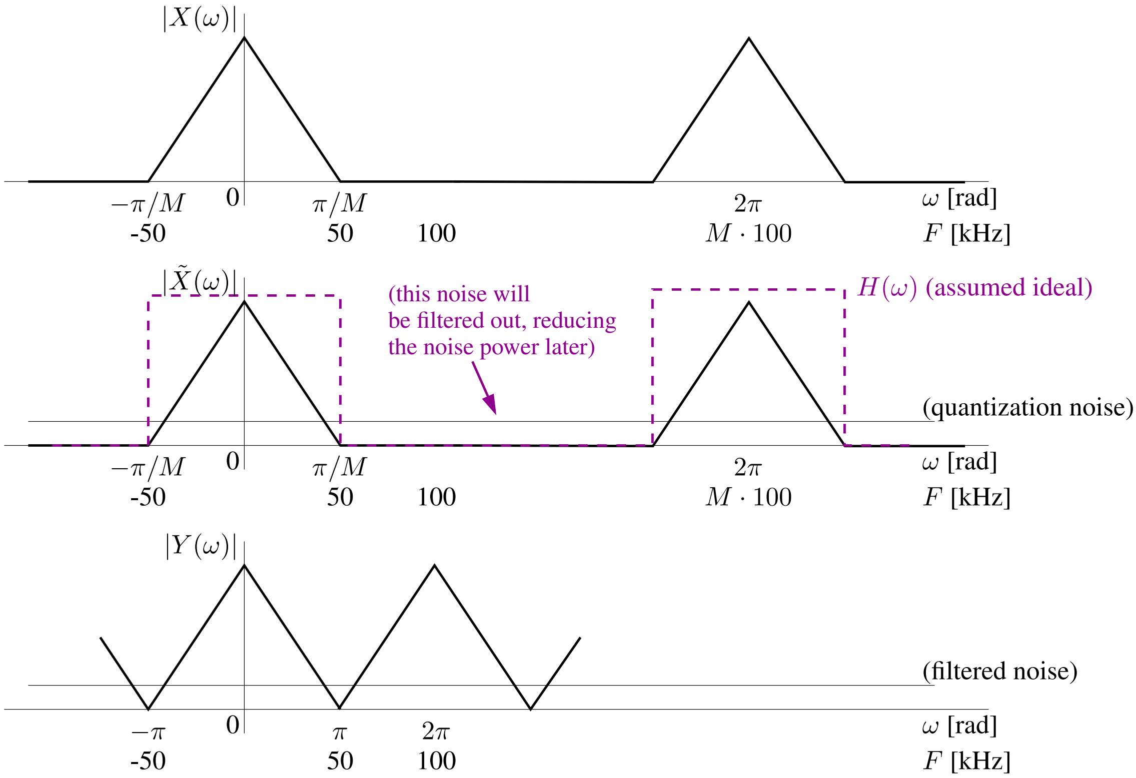
(Drawn for $M = 32$)

1p (b) A lowpass anti-aliasing filter for the downsampling. Ideally, it cuts off at $\omega = \pi/M$, the given non-ideal filter has its 3dB point at this frequency.

1p (c) This noise is also filtered, and we will lose a lot of noise (ideally, a fraction $1/M$ is retained).

Alternatively, we can say that the noise is being averaged ($H(z)$ sums 256 subsequent samples), this will reduce the variance of the noise by a factor M (assuming the noise is white).

2p (d)

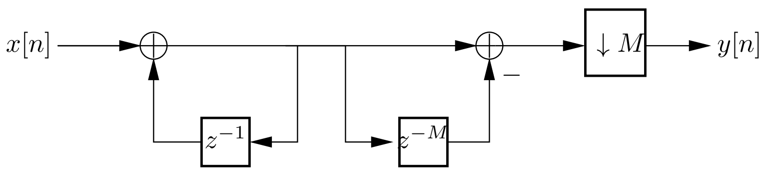


2p (e) Initially, the variance of the quantization noise was $\frac{\Delta^2}{12} = \frac{2^{-2(B+1)}}{12}$, where $B = 4$ bits.

Due to the averaging in the filter, the variance is reduced by a factor M: $\frac{2^{-2(B+1)}}{12M}$. If we equate this to $\frac{2^{-2(B'+1)}}{12}$, then the new number of bits becomes

$$\frac{2^{-2(B'+1)}}{12} = \frac{2^{-2(B+1)}}{12M} \quad \Rightarrow \quad -2(B'+1) = -2(B+1) - \log_2 M \quad \Rightarrow \quad B' = B + \frac{1}{2}\log_2 M = 4+4 = 8$$

2p (f) An initial realization could be:



We can bring the downsampler more to the front, this will replace z^{-M} by z^{-1}.

(A derivation of $H(z)$ using the formulas for upsampling/downsampling is more tricky, e.g. because this is not an LTI system: an expression like $Y(z) = H(z)X(z)$ is not valid.)

Question 3 (7 points)

The joint probability density function of two variables X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} c\, e^{-2x}\, e^{-3y} & \text{for } 0 \le x \le y \le \infty \\ 0 & \text{otherwise.} \end{cases}$$

(a) Calculate the value of constant c.

(b) Calculate the probability $\mathrm{P}[X > 3]$.

(c) Show that the marginal pdf $f_X(x)$ equals

$$f_X(x) = \frac{1}{3}\,c\,e^{-5x}\,, \qquad x \geq 0\,.$$

(d) Use the Chebyshev inequality to find an estimate for $\mathrm{P}[X > 3]$.

(e) Calculate the conditional pdf $f_{X|Y}(x|y)$ and the maximum a posteriori estimator $\hat{X}_{MAP}(Y)$.

(f) Argue whether or not X and Y are independent.

Solution:

1p (a)

$$\begin{aligned} c\int_0^\infty\int_0^y e^{-2x}e^{-3y}\mathrm{d}x\,\mathrm{d}y &= c\int_0^\infty e^{-3y}\left(\frac{1}{2}-\frac{1}{2}e^{-2y}\right)\mathrm{d}y \\ &= c\left(\frac{1}{6}-\frac{1}{10}\right) \\ &= \frac{1}{15}c \quad = \quad 1 \end{aligned}$$

Therefore, $c = 15$. Alternatively, you can first integrate over y like this:

$$c\int_0^\infty\int_x^\infty e^{-2x}e^{-3y}\mathrm{d}y\,\mathrm{d}x = \cdots$$

1p (b)

$$\begin{aligned} \mathrm{P}[X > 3] &= c\int_3^\infty\int_x^\infty e^{-2x}e^{-3y}\mathrm{d}y\,\mathrm{d}x \\ &= c\int_3^\infty e^{-2x}\left(\frac{1}{3}e^{-3x}\right)\mathrm{d}x \\ &= \frac{c}{15}e^{-15} \quad = \quad e^{-15} \quad \approx \quad 3\cdot 10^{-7} \end{aligned}$$

Alternatively, you can first integrate over x like this:

$$\mathrm{P}[X > 3] = c\int_3^\infty\int_3^y e^{-2x}e^{-3y}\mathrm{d}x\,\mathrm{d}y = \cdots$$

Alternatively, you can use $f_X(x)$ (given in item c).

1p (c) For $x > 0$,

$$f_X(x) = c\int_x^\infty e^{-2x}e^{-3y}\mathrm{d}y = \frac{c}{3}e^{-5x} = 5e^{-5x}$$

1.5p (d) It is clear that $f_X(x)$ has an exponential distribution with $\lambda = 5$. Therefore (see table) $\mathrm{E}[X] = 1/5$ and $\mathrm{var}[X] = 1/25$.

$$\begin{aligned} \mathrm{P}[X > 3] &= \mathrm{P}[X - \mathrm{E}[X] > 3 - \mathrm{E}[X]] \\ &\leq \mathrm{P}[|X - \mathrm{E}[X]| > 3 - \mathrm{E}[X]] \\ &\leq \frac{\mathrm{var}[X]}{(3 - \mathrm{E}[X])^2} \\ &= \frac{1/25}{(3 - 1/5)^2} \quad \approx \quad 5\cdot 10^{-3} \end{aligned}$$

2p (e)

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

We can compute $f_Y(y)$ as

$$f_Y(y) = \int_0^y f_{X,Y}(x,y)\mathrm{d}x = \cdots = \frac{c}{2}\left(e^{-3y} - e^{-5y}\right) \quad (0 \leq y \leq \infty)$$

so that

$$f_{X|Y}(x|y) = \begin{cases} 2e^{-2x}\,\frac{1}{1-e^{-2y}} & \text{for } 0 \leq x \leq y \\ 0 & \text{otherwise} \end{cases}$$

The MAP estimator is

$$\hat{x}_{MAP}(y) = \arg\max_x f_{X|Y}(x|y) = \arg\max_{x,\,0\leq x\leq y}\ 2e^{-2x}\,\frac{1}{1-e^{-2y}} = 0$$

which is in fact independent of the observation y.

(Observe that the precise expression for $f_Y(y)$ is actually not needed as this is just a function of y and we will only optimize over x.)

.5p (f) X and Y are not independent, e.g. because of the condition $x \leq y$.

Question 4 (6 points)

We consider the non-stationary stochastic process $X(t)$

$$X(t) = \begin{cases} At^2 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

where A is a random variable with the following uniform distribution:

$$f_A(a) = \begin{cases} c & \text{for } 0 \leq a \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Plot three different realizations of this process.

(b) Characterize this process: (1) Is it continuous-value or discrete-value? (2) Is it continuous-time or discrete-time? (3) Is it stationary?

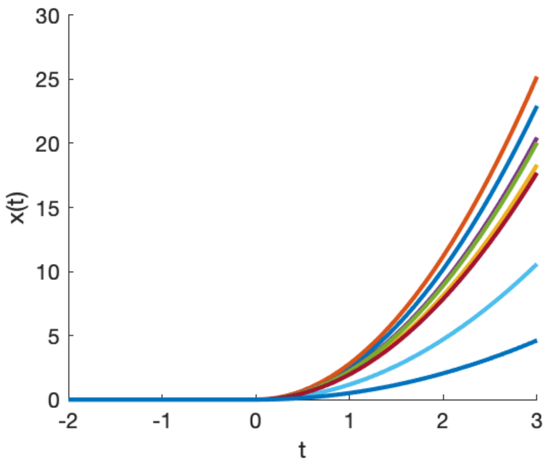
(c) Specify the pdf $f_{X(t)}(x)$ and determine the value of the constant c.

(d) Calculate the expected value $\mathrm{E}[X(t)]$.

(e) Calculate the autocorrelation function $R_X(t,\tau)$.

Solution:

1p (a) Here are 8 realizations:



1p (b) Continuous-time, continuous-value, not stationary.

2p (c) The pdf of A is a uniform pdf and should integrate to 1, therefore $c = 1/3$.

For $t < 0$, the value of $X(t)$ is always 0, so in this case the pdf is $\delta(x)$. Otherwise, for some $t \geq 0$, the value of $X(t)$ is uniform between 0 and $3t^2$. Thus,

$$f_{X(t)}(x) = \begin{cases} \frac{1}{3t^2} & t \geq 0,\ 0 \leq x \leq 3t^2 \\ \delta(x) & t < 0 \\ 0 & \text{otherwise} \end{cases}$$

1p (d) For $t \geq 0$,

$$\mathrm{E}[X(t)] = \begin{cases} t^2\mathrm{E}[A] = \frac{3}{2}t^2 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Alternatively, you can integrate $f_{X(t)}(x)$ but you have to be careful not to integrate over t:

$$t \geq 0: \qquad \mathrm{E}[X(t)] = \int x f_{X(t)}(x)\mathrm{d}x = \int_0^{3t^2} \frac{x}{3t^2}\mathrm{d}x = \left[\frac{x^2}{2 \cdot 3t^2}\right]_0^{3t^2} = \frac{3}{2}t^2$$

1p (e) Since $\mathrm{E}[A^2] = \int_0^3 a^2/3\mathrm{d}a = 3$,

$$R_X(t,\tau) = \mathrm{E}[X(t)X(t+\tau)] = \begin{cases} \mathrm{E}[A^2t^2(t+\tau)^2] = 3t^2(t+\tau)^2 & t \geq 0 \text{ and } t+\tau \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

Question 5 (3 points)

We consider an LTI system with input process $X(t)$, output process $Y(t)$, and impulse response $h(t)$ given by

$$h(t) = \begin{cases} 4e^{-2t} & \text{for } t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

The input process $X(t)$ is WSS, zero mean, uncorrelated with variance σ_X^2.

(a) Calculate the autocorrelation function $R_Y(\tau)$ of the output.

(b) Calculate the power spectral density $S_Y(f)$ of the output.

Solution:

2p (a) We have $R_Y(\tau) = R_X(\tau) * h(\tau) * h(-\tau)$, with $R_X(\tau) = \sigma_X^2 \delta(\tau)$. First calculate $g(\tau) = h(\tau) * h(-\tau)$:

$$\begin{aligned} g(\tau) &= \int h(t)h(-\tau+t)\mathrm{d}t \\ &= \int_{-\infty}^{\infty} 4e^{-2t}u(t)4e^{-2t+2\tau}u(-\tau+t)\mathrm{d}t \\ &= \begin{cases} 16e^{2\tau}\int_{\tau}^{\infty} e^{-4t}\mathrm{d}t = 4e^{-2\tau} & \text{for } \tau \geq 0 \\ 16e^{2\tau}\int_{0}^{\infty} e^{-4t}\mathrm{d}t = 4e^{2\tau} & \text{for } \tau < 0 \end{cases} \\ &= 4e^{-2|\tau|} \end{aligned}$$

Then

$$R_Y(\tau) = R_X(\tau) * g(\tau) = 4\sigma_X^2\, e^{-2|\tau|}\,.$$

1p (b) We can take the Fourier Transform of $R_Y(\tau)$ (use the table):

$$S_Y(f) = 2\sigma_X^2 \frac{8}{4+(2\pi f)^2}$$

Alternatively, use $H(f) = \frac{4}{2+j2\pi f}$ (use the table) and $S_X(f) = \sigma_X^2$ so that

$$S_Y(f) = |H(f)|^2 S_X(f) = \frac{4}{2+j2\pi f}\cdot\frac{4}{2-j2\pi f}\sigma_X^2 = \frac{16}{4+(2\pi f)^2}\sigma_X^2$$