Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Signal Processing Systems

Resit exam EE2S31 SIGNAL PROCESSING
16 July 2025 13:30–16:30

Closed book; two sides of one A4 with handwritten notes permitted. No other tools except a basic pocket calculator permitted. Note the attached tables!

This exam consists of five questions (30 points). Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.

Question 1 (5 points)

An audio signal $x_a(t)$ is sampled at 40 kHz such that we obtain a sequence $x[n] = x_a(nT)$. We take samples during 2.5 seconds. The signal has to be filtered by an FIR filter $h[n]$ with 1024 coefficients.

(a) What is the highest admissible frequency in the signal $x_a(t)$?

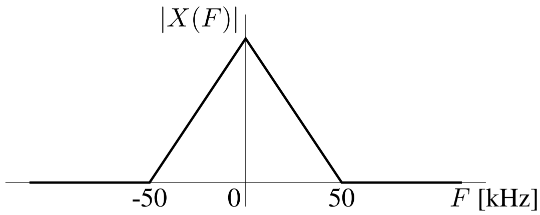
(b) How many multiplications are needed to apply the filter to the data $x[n]$, using a direct implementation of the convolution? (Ignore side effects in this calculation.)

(c) We would like to use the FFT to reduce the number of operations. Following the overlap-add method, describe in detail how the convolution can be implemented using the FFT. Also give a block scheme.

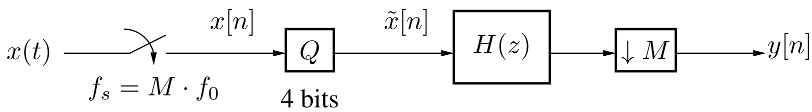
(d) How many multiplications are needed now? [Assume that the FFT of order N has a complexity of $N \log_2(N)$.]

Question 2 (9 points)

A signal $x(t)$ has an amplitude spectrum schematically shown as follows:



To sample $x(t)$, an oversampling AD converter operates at $M = 256$ times the desired sampling rate $f_0 = 100$ kHz, and quantizes samples at 4 bits. Digitally, the sample rate is reduced by a factor $M = 256$. See figure:



$H(z)$ is a lowpass filter given by $H(z) = 1 + z^{-1} + \cdots + z^{-(M-1)} = \dfrac{1 - z^{-M}}{1 - z^{-1}}$.

(a) Draw the amplitude spectrum $|H(e^{j\omega})|$. (Clearly label the frequency axis. For illustration purposes, you may consider a small M.)

In this plot, what is the location (value of ω) of the first zero of the transfer function?

(b) What is the role of this filter in the context of the oversampling ADC architecture?

If it was an ideal filter, what would be its specification?

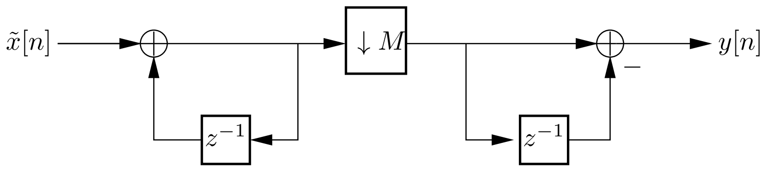
(c) What is the effect of this filter on the quantization noise?

(d) Draw amplitude spectra for the signals $x[n]$, $\tilde{x}[n]$, and $y[n]$. Label the frequency axis both in terms of ω [rad] and the corresponding original frequencies F [kHz].

(e) As usual, we model the effect of the quantization as additive white noise, with a uniform distribution. Its variance is $\sigma_e^2 = \Delta^2/12$ with $\Delta = R/2^{B+1}$ where R is the range of the quantizer and $B = 4$ is the number of bits. You may consider $R = 1$.

How many bits accuracy do you expect at the output? (Give a derivation.)

(f) An efficient implementation of the filter, in combination with the downsampler, is as follows:



Prove that this implementation indeed results in the desired response.

(*Hint:* recall the "noble identities".)

Question 3 (7 points)

The joint probability density function of two variables X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} c\, e^{-2x}\, e^{-3y} & \text{for } 0 \le x \le y \le \infty \\ 0 & \text{otherwise.} \end{cases}$$

(a) Calculate the value of constant c.

(b) Calculate the probability $\mathrm{P}[X > 3]$.

(c) Show that the marginal pdf $f_X(x)$ equals

$$f_X(x) = \frac{1}{3}\, c\, e^{-5x}\,, \qquad x \ge 0\,.$$

(d) Use the Chebyshev inequality to find an estimate for $\mathrm{P}[X > 3]$.

(e) Calculate the conditional pdf $f_{X|Y}(x|y)$ and the maximum a posteriori estimator $\hat{X}_{MAP}(Y)$.

(f) Argue whether or not X and Y are independent.

Question 4 (6 points)

We consider the non-stationary stochastic process $X(t)$

$$X(t) = \begin{cases} At^2 & \text{for } t \ge 0 \\ 0 & \text{for } t < 0 \end{cases}$$

where A is a random variable with the following uniform distribution:

$$f_A(a) = \begin{cases} c & \text{for } 0 \le a \le 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Plot three different realizations of this process.

(b) Characterize this process: (1) Is it continuous-value or discrete-value? (2) Is it continuous-time or discrete-time? (3) Is it stationary?

(c) Specify the pdf $f_{X(t)}(x)$ and determine the value of the constant c.

(d) Calculate the expected value $\mathrm{E}[X(t)]$.

(e) Calculate the autocorrelation function $R_X(t,\tau)$.

Question 5 (3 points)

We consider an LTI system with input process $X(t)$, output process $Y(t)$, and impulse response $h(t)$ given by

$$h(t) = \begin{cases} 4e^{-2t} & \text{for } t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

The input process $X(t)$ is WSS, zero mean, uncorrelated with variance σ_X^2.

(a) Calculate the autocorrelation function $R_Y(\tau)$ of the output.

(b) Calculate the power spectral density $S_Y(f)$ of the output.

Tables from Stochastic Processes, Appendix A

Exponential (λ)

For $\lambda > 0$,

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad \phi_X(s) = \frac{\lambda}{\lambda - s}$$

$$\mathrm{E}[X] = 1/\lambda$$

$$\mathrm{Var}[X] = 1/\lambda^2$$

Uniform (a, b)

For constants $a < b$,

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases} \qquad \phi_X(s) = \frac{e^{bs} - e^{as}}{s(b-a)}$$

$$\mathrm{E}[X] = \frac{a+b}{2}$$

$$\mathrm{Var}[X] = \frac{(b-a)^2}{12}$$

Gaussian (μ, σ)

For constants $\sigma > 0$, $-\infty < \mu < \infty$,

$$f_X(x) = \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sigma\sqrt{2\pi}} \qquad \phi_X(s) = e^{s\mu + s^2\sigma^2/2}$$

$$\mathrm{E}[X] = \mu$$

$$\mathrm{Var}[X] = \sigma^2$$

Tables from Stochastic Processes (R.D. Yates and D.J. Goodman):

Time function	Fourier Transform
$\delta(\tau)$	1
1	$\delta(f)$
$\delta(\tau-\tau_0)$	$e^{-j2\pi f\tau_0}$
$u(\tau)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$
$e^{j2\pi f_0\tau}$	$\delta(f-f_0)$
$\cos 2\pi f_0\tau$	$\frac{1}{2}\delta(f-f_0)+\frac{1}{2}\delta(f+f_0)$
$\sin 2\pi f_0\tau$	$\frac{1}{2j}\delta(f-f_0)-\frac{1}{2j}\delta(f+f_0)$
$ae^{-a\tau}u(\tau)$	$\frac{a}{a+j2\pi f}$
$ae^{-a\|\tau\|}$	$\frac{2a^2}{a^2+(2\pi f)^2}$
$ae^{-\pi a^2\tau^2}$	$e^{-\pi f^2/a^2}$
$\mathrm{rect}(\tau/T)$	$T\,\mathrm{sinc}(fT)$
$\mathrm{sinc}(2W\tau)$	$\frac{1}{2W}\,\mathrm{rect}(\frac{f}{2W})$

Note that a is a positive constant and that the rectangle and sinc functions are defined as

$$\mathrm{rect}(x)=\begin{cases}1 & |x|<1/2,\\ 0 & \text{otherwise},\end{cases}$$

$$\mathrm{sinc}(x)=\frac{\sin(\pi x)}{\pi x}.$$

Table 1 Fourier transform pairs of common signals.

Time function	Fourier Transform
$g(\tau-\tau_0)$	$G(f)e^{-j2\pi f\tau_0}$
$g(\tau)e^{j2\pi f_0\tau}$	$G(f-f_0)$
$g(-\tau)$	$G^*(f)$
$\frac{dg(\tau)}{d\tau}$	$j2\pi fG(f)$
$\int_{-\infty}^{\tau} g(v)\,dv$	$\frac{G(f)}{j2\pi f}+\frac{G(0)}{2}\delta(f)$
$\int_{-\infty}^{\infty} h(v)g(\tau-v)\,dv$	$G(f)H(f)$
$g(t)h(t)$	$\int_{-\infty}^{\infty} H(f')G(f-f')\,df'$

Table 2 Properties of the Fourier transform.