Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Signal Processing Systems

Resit exam EE2S31 SIGNAL PROCESSING
22 April 2025 13:30–16:30

Closed book; two sides of one A4 with handwritten notes permitted. No other tools except a basic pocket calculator permitted. Note the attached tables!

This exam consists of five questions (30 points). Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.

Question 1 (6 points)

Let $x[n], n = 0, \cdots, N-1$ be a series of N samples, and assume $X[k]$ is its DFT.

(a) Show that if $x[n]$ is a symmetric series (that is, $x[n] = x[N-1-n]$), and N is even, then $X[N/2] = 0$.

(b) Show that if $x[n] = -x[n+M]$ with $N = 2M$, then $X[2\ell] = 0$ for $\ell = 0, 1, \ldots, M-1$.

(c) Let us assume that $y[n]$ is an extension of $x[n]$ with $(M-1)N$ zeros till length MN (i.e., zero padding), and $Y[k]$ is its MN-point DFT. Show that $Y[k]$ interpolates $X[k]$, i.e.,

$$X[k] = Y[kM], \quad 0 \le k \le N-1\,.$$

Solution

Using the definition: $X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$.

2p (a)

$$X[\frac{N}{2}] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}\frac{N}{2}n} = \sum_{n=0}^{N-1} x[n] e^{-j\pi n} = \sum_{n=0}^{N-1} x[n](-1)^n = \sum_{n=0}^{N/2-1} x[n](-1)^n + \sum_{n=N/2}^{N-1} x[n](-1)^n$$

The second term can be written as

$$\sum_{n=N/2}^{N-1} x[n](-1)^n = \sum_{n=N/2}^{N-1} x[N-1-n](-1)^n = \sum_{r=0}^{N/2-1} -x[r](-1)^r$$

In the last step we substituted $n = N-1-r \Leftrightarrow r = N-1-n$, the limits of the summation are, respectively, $r = N/2-1$ and $r = 0$.

2p (b)

$$\begin{aligned}
X[2\ell] &= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}(2\ell)n} \\
&= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{M}\ell n} \\
&= \sum_{n=0}^{M-1} x[n] e^{-j\frac{2\pi}{M}\ell n} + \sum_{n=M}^{2M-1} x[n] e^{-j\frac{2\pi}{M}\ell n} \\
&= \sum_{n=0}^{M-1} x[n] e^{-j\frac{2\pi}{M}\ell n} + \sum_{r=0}^{M-1} x[r+M] e^{-j\frac{2\pi}{M}\ell(r+M)} \\
&= \sum_{n=0}^{M-1} x[n] e^{-j\frac{2\pi}{M}\ell n} - \sum_{r=0}^{M-1} x[r] e^{-j\frac{2\pi}{M}\ell r} \\
&= 0
\end{aligned}$$

We substituted $r = n - M \Leftrightarrow n = r + M$.

2p (c)

$$\begin{aligned} Y[kM] &= \sum_{n=0}^{MN-1} y[n] e^{-j\frac{2\pi}{MN}(kM)n} \\ &= \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn} \\ &= X[k]. \end{aligned}$$

Question 2 (4 points)

Let us consider the sampling of an audio signal. The amplitude of the signal reaches ± 2V. We sample the signal at $F_s = 40$ kHz with an A/D converter with a range of $\begin{bmatrix} -3 & 3 \end{bmatrix}$V and using 3 bits plus a sign bit. After sampling, we observe that the autocorrelation of the samples at lag 1 is $r_{xx}[1] = 0.4$. We further observe that the signal-to-quantization-ratio (SQNR) is too low. Which of the following options can we use to increase the SQNR with at least 6 dB? Answer 'yes', 'no', or 'not enough information' and motivate your answer for each option.

(a) Add an extra bit to the quantizer

(b) Decrease the range of the quantizer

(c) Differential quantization

(d) Predictive quantization

Solution

$\text{SQNR} = 10 \log \frac{\sigma_x^2}{\sigma_e^2} = \cdots = 6.02\, b + 16.81 - 20 \log \left(\frac{R}{\sigma_x} \right)$ dB, where b is the number of bits (excluding sign bit), R is the range of the quantizer, and σ_x is the input signal variance.

1p (a) Yes. An extra bit will increase SQNR with approximately 6 dB based on the above formula.

1p (b) No. We can decrease the range to $\begin{bmatrix} -2 & 2 \end{bmatrix}$V according to the amplitude of the input signal. However, this will only yield a 3 dB improvement based on the above formula.

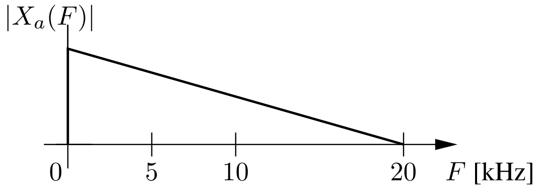
1p (c) No. Differential quantization is only beneficial if $r_{xx}[1] > 0.5$.

1p (d) Not enough information. $\sigma_d^2 = \sigma_x^2(1 - a^2)$, where $a = \frac{r_{xx}[1]}{\sigma_x^2}$. Therefore, the answer depends on the variance of the input signal, which is not given.

Question 3 (8 points)

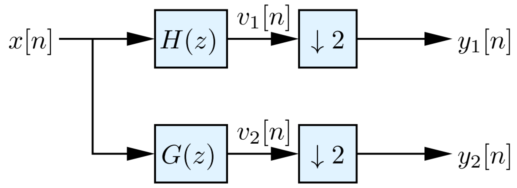
MPEG encoding for audio is based on splitting a frequency band into two halves (L and H); each half can then be encoded with a different number of samples. The low-frequency band is again split into two halves (LL and LH), and this process can be repeated further. In this way, a logarithmically divided frequency spectrum is obtained, which is more natural for audio. It is essential that this splitting can be done efficiently and that the original signal can also be reconstructed.

Let us assume that we sample an audio signal $x_a(t)$ with a sampling frequency $F_s = 40$ kHz. Its spectrum $X_a(F)$ is shown below:



Let us further assume that $H(z)$ is an ideal FIR lowpass filter with a cutoff frequency ω_c. The impulse response $h[n]$ has N coefficients $h_0, \cdots, h_{N-1}$.

We use $H(z)$ as an anti-aliasing filter before downsampling with a factor $M = 2$, as shown in the upper part of the following schematic:



(a) What is the highest possible value for ω_c for which no aliasing occurs during downsampling?

(b) For this case, draw the spectra of $x[n]$, $v_1[n]$ and $y_1[n]$.

We define a filter $G(z)$ with impulse response $g[n] = (-1)^n h[n]$, and use this $G(z)$ in combination with downsampling with a factor of 2.

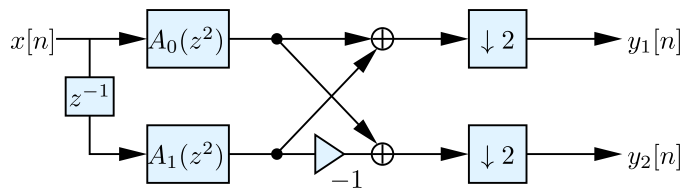
(c) Show that $g[n]$ is a highpass filter. What is the cutoff frequency in terms of the previously defined ω_c?

(d) Draw the spectra of $v_2[n]$ and $y_2[n]$.

(e) We write $H(z) = A_0(z^2) + z^{-1}A_1(z^2)$, where $A_0(z)$ and $A_1(z)$ are polynomials with $N/2$ coefficients (we assume here that N is even).

What are $A_0(z)$ and $A_1(z)$ in terms of the coefficients of $h[n]$?

(f) Explain why the following schematic is a correct realization for both $H(z)$ and $G(z)$.



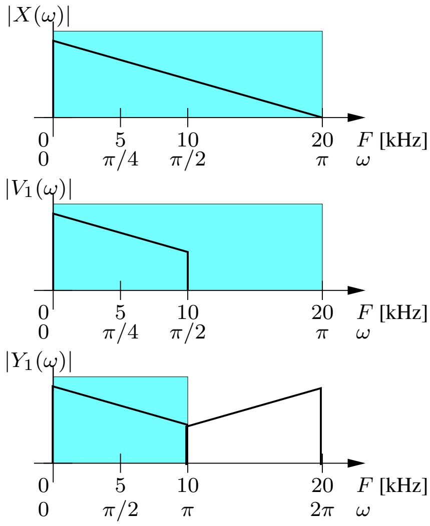
(g) How can you improve the computational efficiency of the above realization?

Hint (time and frequency shifting properties of DTFT): If DTFT of $x[n]$ is $X(\omega)$, then the DTFT of $x[n - n_0]$ is $e^{-j\omega n_0}X(\omega)$ and the DTFT of $x[n]e^{j\omega_0 n}$ is $X(\omega - \omega_0)$.

Solution

1p (a) $\omega_c = \pi/2$ (the filter selects half the band). This corresponds to $F_c = 10$ kHz.

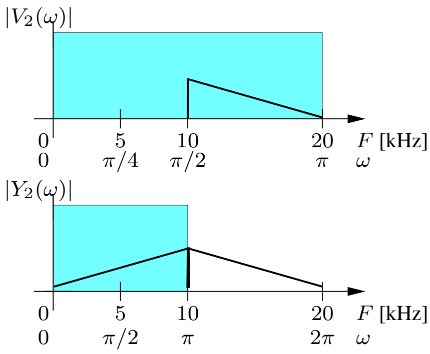
1.5p (b)



1.5p (c) Multiplication (modulation) with $e^{j\omega_0 n} = (-1)^n$ in time domain corresponds to shifting by $\omega_0 = \pi$ in frequency domain. Hence, the spectrum of the lowpass filter $H(e^{j\omega})$ is shifted by π and it becomes a highpass filter.

The cutoff frequency of $-\omega_c = -\pi/2$ is shifted to $\omega_c' = \pi/2$.

1p (d)



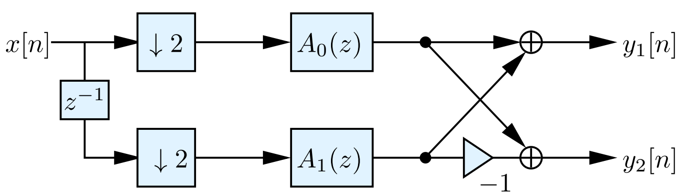
1p (e) $A_0(z) = h_0 + h_2 z^{-1} + h_4 z^{-2} + \cdots$, and $A_1(z) = h_1 + h_3 z^{-1} + h_5 z^{-2} + \cdots$.

1p (f) We have $g[n] = (-1)^n h[n]$, the odd coefficients become negative. These are all in $A_1(z)$. Hence

$$\begin{aligned} H(z) &= A_0(z^2) + z^{-1} A_1(z^2) \\ G(z) &= A_0(z^2) - z^{-1} A_1(z^2) \end{aligned}$$

These are exactly the relations shown in the schematic.

1p (g) The two downsamplers can both be moved in front of the A_0 and A_1 filters, which will turn into $A_0(z)$ and $A_1(z)$. These filters run at half the rate, which saves a factor 2 on computations.



Question 4 (7 points)

The random variables X and Y have the joint probability density function (pdf)

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2} & \text{for } -1 \le x \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that the marginal pdfs $f_X(x)$ and $f_Y(y)$ are given by

$$f_X(x) = \frac{1}{2} - \frac{x}{2} \quad \text{for } -1 \le x \le 1$$

and

$$f_Y(y) = \frac{y}{2} + \frac{1}{2} \quad \text{for } -1 \le y \le 1\,.$$

(b) Calculate $\mathrm{E}[X|X \ge 0]$.

(c) Calculate the MMSE estimator $\hat{X} = \mathrm{E}[X|Y]$.

(d) Determine the correlation $\mathrm{E}[XY]$.

(e) Argue whether or not X and Y are: 1) orthogonal, 2) correlated and 3) independent.

Solution:

1.5p (a)

$$f_X(x) = \int_{y=x}^{y=1} \frac{1}{2}\mathrm{d}y = \left[\frac{y}{2}\right]_{y=x}^{y=1} = \frac{1}{2} - \frac{x}{2} \quad \text{for } -1 \le x \le 1\,.(0\,\text{otherwise})$$

$$f_Y(y) = \int_{x=-1}^{x=y} \frac{1}{2}\mathrm{d}x = \left[\frac{x}{2}\right]_{x=-1}^{x=y} = \frac{y}{2} + \frac{1}{2} \quad \text{for } -1 \le y \le 1\,.(0\,\text{otherwise})$$

2p (b)

$$\mathrm{P}[X \ge 0] = \int_{x=0}^{x=1} (\frac{1}{2} - \frac{x}{2})\mathrm{d}x = \left[\frac{x}{2} - \frac{x^2}{4}\right]_{x=0}^{x=1} = \frac{1}{4}$$

$$f_{X,X\ge 0}(x) = \begin{cases} \frac{1}{\frac{1}{4}}\left(\frac{1}{2} - \frac{x}{2}\right) = 2(1-x) & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathrm{E}[X|X \ge 0] = \int_0^1 x\,2(1-x)\mathrm{d}x = \left[x^2 - \frac{2}{3}x^3\right]_0^1 = \frac{1}{3}\,.$$

1p (c)

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{1}{y+1} & -1 \le x \le y \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{E}[X|Y=y] = \int x f_{X|Y}(x|y)\mathrm{d}x = \int_{-1}^{y} x\frac{1}{y+1}\mathrm{d}x = \left[\frac{x^2}{2}\frac{1}{y+1}\right]_{-1}^{y} = \frac{y^2-1}{2(y+1)} = \frac{1}{2}(y-1)$$

Hence $\hat{X}(Y) = \frac{1}{2}(Y-1)$.

1p (d)

$$\mathrm{E}[XY] = \int_{-1}^{1}\int_{x}^{1} \frac{xy}{2}\mathrm{d}y\mathrm{d}x = \int_{-1}^{1}\left[\frac{xy^2}{4}\right]_x^1 \mathrm{d}x = \int_{-1}^{1}\left[\frac{x}{4} - \frac{x^3}{4}\right]\mathrm{d}x = \left[\frac{x^2}{8} - \frac{x^4}{16}\right]_{-1}^{1} = 0\,.$$

1.5p (e) 1) X and Y are orthogonal, as the correlation $\mathrm{E}[XY] = 0$.

2) $\mathrm{E}[X] = -1/3$ and $\mathrm{E}[Y] = 1/3$. Therefore, the covariance is $\mathrm{Cov}[X,Y] = \mathrm{E}[XY] - \mathrm{E}[X]\,\mathrm{E}[Y] = 1/9$, and hence, X and Y are correlated.

3) X and Y are not independent, as $f_X f_Y \neq f_{X,Y}$.

The fact that X and Y are correlated also implies that they are not independent.

Question 5 (5 points)

We consider the non-stationary stochastic process $X(t)$ with pdf

$$f_{X(t)}(x) = \begin{cases} \frac{1}{3t} & \text{for } 3t \leq x \leq 6t \text{ and } t > 0 \\ \delta(x) & \text{for } t \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Calculate $\mathrm{E}[X(t)]$ for $t > 0$, and show that it equals $\mathrm{E}[X(t)] = 4.5t$.

Given is a linear time invariant system with impulse response

$$h(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

The input to this system is process $X(t)$, and the output is $Y(t)$.

(b) Calculate $\mathrm{E}[Y(t)]$ for $t > 0$.

Now we consider a different system with input process $U(t)$ and output process $V(t)$, and impulse response $g(t)$ given by

$$g(t) = \begin{cases} 1 & \text{for } 1 \leq t \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

The input process $U(t)$ is WSS and its autocorrelation function is given by $R_U(\tau) = \sigma^2\delta(\tau)$.

(c) Calculate the cross-correlation $R_{UV}(\tau)$ between input and output.

(d) Draw the autocorrelation $R_V(\tau)$ of output $V(t)$. (Mark the values on both axes.)

(e) Calculate the power spectral density $S_V(f)$ of the output.

Solution:

1p (a) For $t > 0$ we have

$$\mathrm{E}[X(t)] = \int_{3t}^{6t} \frac{x}{3t}\mathrm{d}x = \left[\frac{x^2}{6t}\right]_{3t}^{6t} = \frac{36t^2 - 9t^2}{6t} = 4.5t$$

1p (b) For $t > 0$:

$$\mathrm{E}[Y(t)] = \int_{-\infty}^{\infty} h(u)\mathrm{E}\left[X(t-u)\right]\mathrm{d}u = \int_0^1 4.5(t-u)\mathrm{d}u = \frac{9}{2}t - \frac{9}{4}.$$

1p (c)

$$R_{UV}(\tau) = g(\tau) * R_U(\tau) = \int_1^2 \sigma^2 \delta(\tau - u) \mathrm{d}u = \begin{cases} \sigma^2 & \text{for } 1 \le \tau \le 2 \\ 0 & \text{otherwise} \end{cases}$$

1p (d) We can look at $g(\tau)$ as a time-shifted block shaped impulse reponse around zero, where the time shift does not influence the final correlation. Convolving $g(\tau)$ and $g(-\tau)$ gives a triangular function of height σ^2 and baseline from $\tau = -1$ up to $\tau = +1$. Therefore,

$$R_V(\tau) = g(\tau) * g(-\tau) * R_U(\tau) = \begin{cases} \sigma^2(1 - |\tau|) & \text{for } -1 \le \tau \le 1 \\ 0 & \text{otherwise} \end{cases}$$

1p (e) Recognize $g(t)$ as a rectangular function $\mathrm{rect}(t)$, shifted by $t_0 = 1.5$. Using Table 1, $H(f) = e^{-j2\pi f t_0} \mathrm{sinc}(f)$ with $t_0 = 1.5$, so

$$S_V(f) = |H(f)|^2 S_U(f) = \mathrm{sinc}^2(f)\sigma^2 .$$