Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Signal Processing Systems

Partial exam EE2S31 SIGNAL PROCESSING
Part 1: 22 May 2024 (13:30-15:30)

Closed book; two sides of one A4 with handwritten notes permitted. No other tools except a basic pocket calculator permitted. Note the attached tables!

This exam consists of four questions (34 points). Answer in English. Make clear in your answer how you reach the final result; the road to the answer is very important.

Question 1 (10 points)

Given the joint probability density function of two random variables X and Y:

$$f_{X,Y}(x,y) = \begin{cases} c & 0 \le x \le 3,\ 0 \le y \le \frac{1}{2}x \\ 0 & \text{otherwise.} \end{cases}$$

(a) Calculate the value of the constant c.

(b) Show that $f_X(x)$ and $f_Y(y)$ are given by

$$f_X(x) = \begin{cases} \frac{1}{2}cx & 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} 3c - 2cy & 0 \le y \le \frac{3}{2} \\ 0 & \text{otherwise} \end{cases}$$

and argue whether the random variables X and Y are independent or not.

(c) Determine the blind MMSE estimator $\hat{x}_1$, which is based only on prior information.

(d) Suppose that we have additional knowledge that $X \in A$ with $A = \{X > 2\}$.
Determine the MMSE estimator $\hat{x}_2$ that takes this information into account.

(e) Instead, suppose that we observe a realization y of the random variable Y.
Determine the MMSE estimator $\hat{x}_3(y)$ that uses this information.

Question 2 (8 points)

Last month, newspapers reported that King's Day (27 April) is apparently 1°C colder than Queen's Day (30 April). This was based on observations since 1949.

Denote the temperature on 27 April by the random variable T_K and on 30 April by T_Q. Long-term KNMI statistics suggest the following:

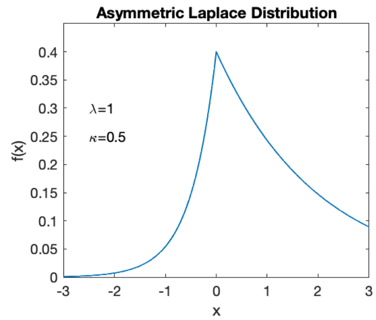
$$\begin{aligned} &\mathrm{E}[T_K] = 14.2, \qquad \mathrm{std}[T_K] = 2.0 \\ &\mathrm{E}[T_Q] = 14.7 \end{aligned}$$

Let Y be the average of the temperatures measured on King's Day over the last 75 years.

(a) Using Chebyshev's inequality, find an upper bound for the probability that Y is at least 1°C colder than $\mathrm{E}[T_Q]$.

(b) Estimate the probability that Y is at least 1°C colder than $\mathrm{E}[T_Q]$ using the central limit theorem.

The statistics on temperature suggest an asymmetric distribution with outliers: hot extremes are more likely than cold extremes. Let's consider an *Asymmetric Laplace Distribution* (ALD) with parameters λ and κ. The Moment Generating Function (MGF) for this distribution is given by



$$\phi(s) = \frac{1}{(1 + s\frac{\kappa}{\lambda})(1 - s\frac{1}{\lambda\kappa})}, \quad \text{ROC: } -\frac{\lambda}{\kappa} < s < \lambda\kappa$$

Consider $T \sim \text{ALD}(\lambda, \kappa)$.

(c) Show that T can be obtained as the difference of two independent exponentially distributed random variables, $T = X_1 - X_2$, where $X_1 \sim \text{Exp}(\lambda\kappa)$ and $X_2 \sim \text{Exp}(\frac{\lambda}{\kappa})$. *See table.*

(d) Determine $\text{E}[T]$ and $\text{var}[T]$.

Question 3 (6 points)

An audio transmitter emits a sine wave $x(t)$ at a frequency between 20 kHz and 23 kHz, described by the equation

$$x(t) = \sin(2\pi F_T t),$$

where F_T is between 20 kHz and 23 kHz. The receiver aims to estimate the frequency of the transmitted signal.

(a) What is the frequency band of interest for the receiver?

(b) Which sampling frequency should the receiver choose if it follows the Nyquist rate?

Suppose the receiver samples the signal at 16 kHz. Let the discrete time Fourier transform (DTFT) of the signal sampled at 16 kHz be $X(f)$, where f is the normalized frequency.

(c) Sketch the spectrum $|X(f)|$ of the received signal for the normalized frequency range $-0.5 \leq f \leq 0.5$ for two cases: when the signal frequency is 20 kHz and when it is 23 kHz.

(d) How does the receiver estimate the transmit frequency using the DTFT $X(f)$? Justify your estimation technique.

Hint: The Fourier transform of $\sin(2\pi F_0 t)$ *is* $X(F) = \frac{j}{2}\delta(F - F_0) - \frac{j}{2}\delta(F + F_0)$.

Question 4 (10 points)

Consider a discrete-time signal $x[n] = [x[0], x[1], x[2], \ldots, x[N-1]]$. Let the DTFT of $x[n]$ be $X(f)$ and its N-point discrete Fourier transform (DFT) be $X[k]$.

Consider another signal $\tilde{x}[n] = [x[N-1], x[0], x[1], \ldots, x[N-2]]$. Let $\tilde{X}[k]$ denote its DFT.

(a) Prove that $\tilde{X}[k] = e^{-j2\pi k/N} X[k]$, for $k = 0, 1, \ldots, N-1$.

Let $y[n]$ be obtained by zero padding $x[n]$ to make its length $2N$, i.e.,

$$y[n] = [x[0], x[1], \ldots, x[N-1], \underbrace{0, 0, \ldots, 0}_{N \text{ zeros}}].$$

Let the DTFT and DFT of $y[n]$ be $Y(f)$ and $Y[k]$, respectively.

(b) Derive a relation between $Y(f)$ and $X(f)$.

(c) What is the relation between the DFTs $X[k]$ and $Y[k]$? Explain your answer.

Consider a filter $h[n] = [h[0], h[1], h[2]]$ and let $N = 32$. Suppose $z[n] = x[n] * h[n]$, where $*$ represents linear convolution.

(d) Explain how to obtain $z[n]$ using the circular convolution.

(e) Explain how to obtain $z[n]$ using the overlap-save method with an 8-point DFT. How many 8-point DFTs and IDFTs are required?

TABLE 4.5 Properties of the Fourier Transform for Discrete-Time Signals

Property	Time Domain	Frequency Domain
Notation	$x(n)$	$X(\omega)$
	$x_1(n)$	$X_1(\omega)$
	$x_2(n)$	$X_2(\omega)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time shifting	$x(n-k)$	$e^{-j\omega k}X(\omega)$
Time reversal	$x(-n)$	$X(-\omega)$
Convolution	$x_1(n) * x_2(n)$	$X_1(\omega)X_2(\omega)$
Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega)$
		$= X_1(\omega)X_2^*(\omega)$
		[if $x_2(n)$ is real]
Wiener–Khintchine theorem	$r_{xx}(l)$	$S_{xx}(\omega)$
Frequency shifting	$e^{j\omega_0 n}x(n)$	$X(\omega - \omega_0)$
Modulation	$x(n)\cos\omega_0 n$	$\frac{1}{2}X(\omega+\omega_0) + \frac{1}{2}X(\omega-\omega_0)$
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega-\lambda)d\lambda$
Differentiation in		
the frequency domain	$nx(n)$	$j\frac{dX(\omega)}{d\omega}$
Conjugation	$x^*(n)$	$X^*(-\omega)$
Parseval's theorem	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega$	

From Appendix A

Exponential (λ)

For $\lambda > 0$,

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad \phi_X(s) = \frac{\lambda}{\lambda - s}$$

$$E[X] = 1/\lambda$$

$$\text{Var}[X] = 1/\lambda^2$$

Table 4.1 The standard normal CDF $\Phi(z)$.

z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$
0.00	0.5000	0.50	0.6915	1.00	0.8413	1.50	0.9332	2.00	0.97725	2.50	0.99379
0.01	0.5040	0.51	0.6950	1.01	0.8438	1.51	0.9345	2.01	0.97778	2.51	0.99396
0.02	0.5080	0.52	0.6985	1.02	0.8461	1.52	0.9357	2.02	0.97831	2.52	0.99413
0.03	0.5120	0.53	0.7019	1.03	0.8485	1.53	0.9370	2.03	0.97882	2.53	0.99430
0.04	0.5160	0.54	0.7054	1.04	0.8508	1.54	0.9382	2.04	0.97932	2.54	0.99446
0.05	0.5199	0.55	0.7088	1.05	0.8531	1.55	0.9394	2.05	0.97982	2.55	0.99461
0.06	0.5239	0.56	0.7123	1.06	0.8554	1.56	0.9406	2.06	0.98030	2.56	0.99477
0.07	0.5279	0.57	0.7157	1.07	0.8577	1.57	0.9418	2.07	0.98077	2.57	0.99492
0.08	0.5319	0.58	0.7190	1.08	0.8599	1.58	0.9429	2.08	0.98124	2.58	0.99506
0.09	0.5359	0.59	0.7224	1.09	0.8621	1.59	0.9441	2.09	0.98169	2.59	0.99520
0.10	0.5398	0.60	0.7257	1.10	0.8643	1.60	0.9452	2.10	0.98214	2.60	0.99534
0.11	0.5438	0.61	0.7291	1.11	0.8665	1.61	0.9463	2.11	0.98257	2.61	0.99547
0.12	0.5478	0.62	0.7324	1.12	0.8686	1.62	0.9474	2.12	0.98300	2.62	0.99560
0.13	0.5517	0.63	0.7357	1.13	0.8708	1.63	0.9484	2.13	0.98341	2.63	0.99573
0.14	0.5557	0.64	0.7389	1.14	0.8729	1.64	0.9495	2.14	0.98382	2.64	0.99585
0.15	0.5596	0.65	0.7422	1.15	0.8749	1.65	0.9505	2.15	0.98422	2.65	0.99598
0.16	0.5636	0.66	0.7454	1.16	0.8770	1.66	0.9515	2.16	0.98461	2.66	0.99609
0.17	0.5675	0.67	0.7486	1.17	0.8790	1.67	0.9525	2.17	0.98500	2.67	0.99621
0.18	0.5714	0.68	0.7517	1.18	0.8810	1.68	0.9535	2.18	0.98537	2.68	0.99632
0.19	0.5753	0.69	0.7549	1.19	0.8830	1.69	0.9545	2.19	0.98574	2.69	0.99643
0.20	0.5793	0.70	0.7580	1.20	0.8849	1.70	0.9554	2.20	0.98610	2.70	0.99653
0.21	0.5832	0.71	0.7611	1.21	0.8869	1.71	0.9564	2.21	0.98645	2.71	0.99664
0.22	0.5871	0.72	0.7642	1.22	0.8888	1.72	0.9573	2.22	0.98679	2.72	0.99674
0.23	0.5910	0.73	0.7673	1.23	0.8907	1.73	0.9582	2.23	0.98713	2.73	0.99683
0.24	0.5948	0.74	0.7704	1.24	0.8925	1.74	0.9591	2.24	0.98745	2.74	0.99693
0.25	0.5987	0.75	0.7734	1.25	0.8944	1.75	0.9599	2.25	0.98778	2.75	0.99702
0.26	0.6026	0.76	0.7764	1.26	0.8962	1.76	0.9608	2.26	0.98809	2.76	0.99711
0.27	0.6064	0.77	0.7794	1.27	0.8980	1.77	0.9616	2.27	0.98840	2.77	0.99720
0.28	0.6103	0.78	0.7823	1.28	0.8997	1.78	0.9625	2.28	0.98870	2.78	0.99728
0.29	0.6141	0.79	0.7852	1.29	0.9015	1.79	0.9633	2.29	0.98899	2.79	0.99736
0.30	0.6179	0.80	0.7881	1.30	0.9032	1.80	0.9641	2.30	0.98928	2.80	0.99744
0.31	0.6217	0.81	0.7910	1.31	0.9049	1.81	0.9649	2.31	0.98956	2.81	0.99752
0.32	0.6255	0.82	0.7939	1.32	0.9066	1.82	0.9656	2.32	0.98983	2.82	0.99760
0.33	0.6293	0.83	0.7967	1.33	0.9082	1.83	0.9664	2.33	0.99010	2.83	0.99767
0.34	0.6331	0.84	0.7995	1.34	0.9099	1.84	0.9671	2.34	0.99036	2.84	0.99774
0.35	0.6368	0.85	0.8023	1.35	0.9115	1.85	0.9678	2.35	0.99061	2.85	0.99781
0.36	0.6406	0.86	0.8051	1.36	0.9131	1.86	0.9686	2.36	0.99086	2.86	0.99788
0.37	0.6443	0.87	0.8078	1.37	0.9147	1.87	0.9693	2.37	0.99111	2.87	0.99795
0.38	0.6480	0.88	0.8106	1.38	0.9162	1.88	0.9699	2.38	0.99134	2.88	0.99801
0.39	0.6517	0.89	0.8133	1.39	0.9177	1.89	0.9706	2.39	0.99158	2.89	0.99807
0.40	0.6554	0.90	0.8159	1.40	0.9192	1.90	0.9713	2.40	0.99180	2.90	0.99813
0.41	0.6591	0.91	0.8186	1.41	0.9207	1.91	0.9719	2.41	0.99202	2.91	0.99819
0.42	0.6628	0.92	0.8212	1.42	0.9222	1.92	0.9726	2.42	0.99224	2.92	0.99825
0.43	0.6664	0.93	0.8238	1.43	0.9236	1.93	0.9732	2.43	0.99245	2.93	0.99831
0.44	0.6700	0.94	0.8264	1.44	0.9251	1.94	0.9738	2.44	0.99266	2.94	0.99836
0.45	0.6736	0.95	0.8289	1.45	0.9265	1.95	0.9744	2.45	0.99286	2.95	0.99841
0.46	0.6772	0.96	0.8315	1.46	0.9279	1.96	0.9750	2.46	0.99305	2.96	0.99846
0.47	0.6808	0.97	0.8340	1.47	0.9292	1.97	0.9756	2.47	0.99324	2.97	0.99851
0.48	0.6844	0.98	0.8365	1.48	0.9306	1.98	0.9761	2.48	0.99343	2.98	0.99856
0.49	0.6879	0.99	0.8389	1.49	0.9319	1.99	0.9767	2.49	0.99361	2.99	0.99861