Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Signal Processing Systems

Partial exam EE2S31 SIGNAL PROCESSING
Part 2: 28 June 2024 (9:00-11:00)

Closed book; two sides of one A4 with handwritten notes permitted. No other tools except a basic pocket calculator permitted. Note the attached tables!

This exam consists of four questions (31 points). Answer in English. Make clear in your answer how you reach the final result; the road to the answer is very important.

Question 1 (8 points)

I sample a cosine wave $x_a(t)$ with a sampling frequency F_s. I obtain the following digital sequence:

$$x[k] = \begin{cases} (-1)^n & \text{if } k = 2n \\ 0 & \text{otherwise} \end{cases}$$

That is, the first 8 samples of $x[k]$ are: $\begin{bmatrix} 1, & 0, & -1, & 0, & 1, & 0, & -1, & 0 \end{bmatrix}$.

The magnitude spectrum of the signal is shown in the following figure:

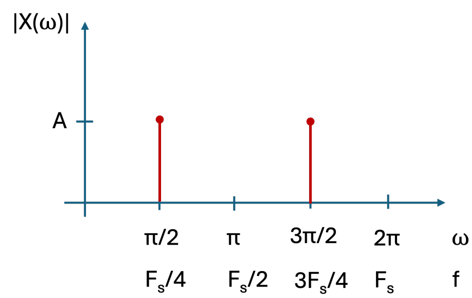


Figure 1.

After interpolating $x[k]$ with a factor $L = 3$, I obtain the signal $x_L[k]$.

(a) Sketch the magnitude spectrum $|X_L(\omega)|$ of the interpolated signal. Make sure you correctly indicate the (normalized) frequencies and the amplitude.

Next, I want to pass the interpolated signal $x_L[k]$ through a filter such that it produces a signal that is equivalent to the signal I would obtain by sampling the original analog signal $x_a(t)$ with rate $3F_s$. Let us denote the filtered signal by $x_{LH}[k]$.

(b) Give the specification of this filter.

I want to convert back my signal to the analog domain with a digital-to-analog converter. After the conversion, I need to use an analog low-pass filter to reject high frequencies (i.e. above π) of the digital spectrum.

(c) Which solution needs an analog filter with a narrower transition band: converting $x[k]$ or converting $x_{LH}[k]$?

(d) Write down the first 12 samples of the interpolated signal $x_L[k]$.

(e) Using the decimation-in-frequency method, compute the 12-point FFT of $x_L[k]$ from the 6-point DFTs of its subsequences.

Hint: As a reminder, the equations for the decimation-in-frequency algorithm are:

$$\begin{aligned} g_1[n] &= x[n] + x[n+\tfrac{N}{2}] \\ g_2[n] &= \left(x[n] - x[n+\tfrac{N}{2}]\right) \cdot W_N^n, \qquad \text{for } n = 0, 1, \cdots, \tfrac{N}{2}-1 \\ X(2k) &= \sum_{n=0}^{(N/2)-1} g_1[n] W_{N/2}^{kn} \\ X(2k+1) &= \sum_{n=0}^{(N/2)-1} g_2[n] W_{N/2}^{kn} \end{aligned}$$

(f) Using the decimation-in-time algorithm depicted in Figure 2 below, compute the 8-point FFT of $x[k]$.

Hint: As verification, you could compare your answer in (e) and (f) with the spectra in Figure 1 and (a).

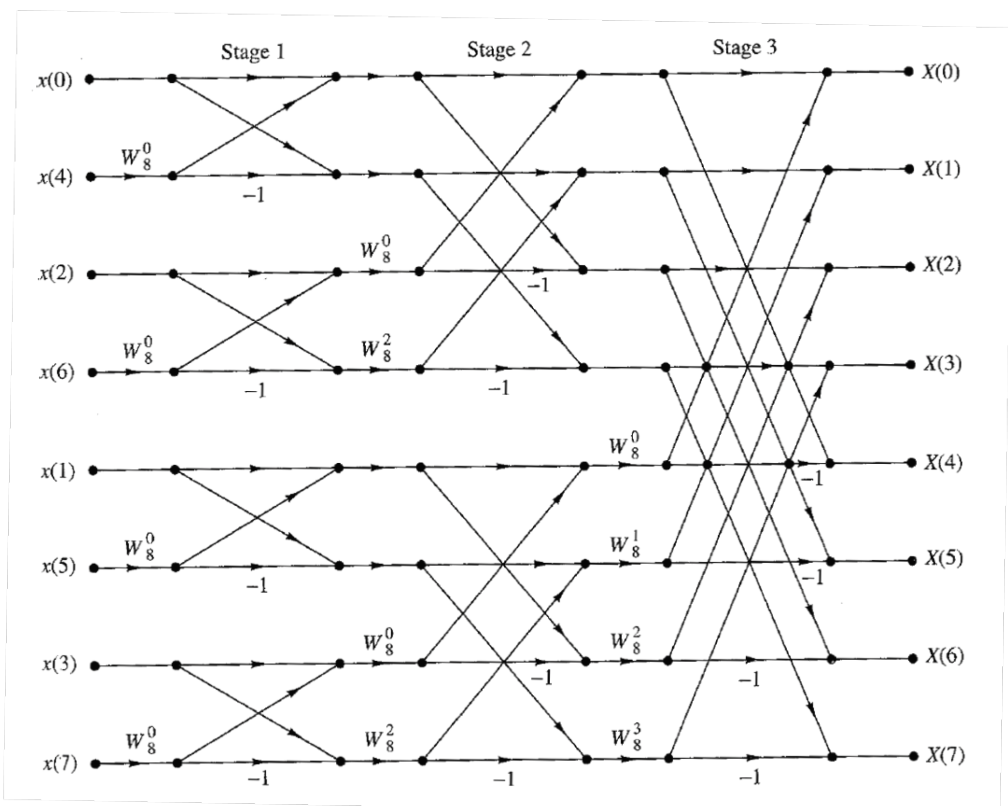
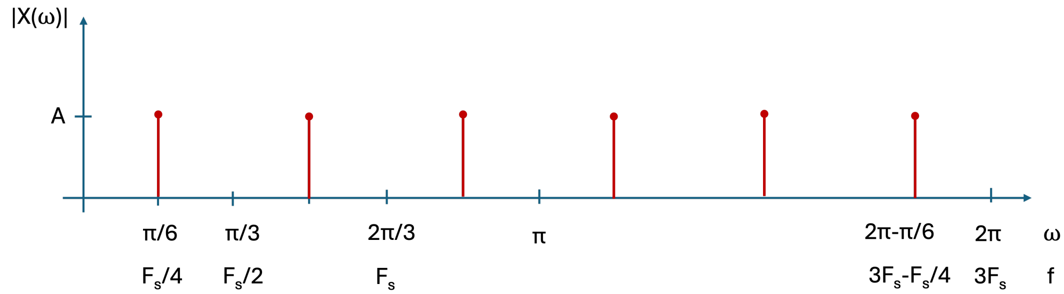


Figure 2. Decimation-in-time FFT

Solution

(a) 1 pnt



(b) 2 pnt Sampling with $3F_s$ would result in a spectrum with an amplitude $3 \cdot A$, and a single spectral line in the fundamental band. Therefore, I need to attenuate the signal and remove the additional 2 copies, with an digital low-pass filter $H(\omega)$:

$$H(\omega) = \begin{cases} 3 & , 0 \leq |\omega| \leq \pi/3 \\ 0 & , \text{otherwise} \end{cases}$$

(Note: the filter above is a design that works in general for $L = 3$ interpolation. In this specific case where we have a line spectrum, other cut-off frequencies are also possible.)

(c) 1 pnt For converting $x_{LH}[k]$ a transition band between $F_s/4$ and $11\,F_s/4$ suffices. Converting $x[k]$ requires a transition band between $F_s/4$ and $3F_s/4$, which is much narrower.

(d) 1 pnt [1, 0 , 0 , 0 , 0 , 0, -1, 0, 0, 0, 0 ,0]

(e) 2 pnt

$$g_1[n] = \begin{bmatrix}1, & 0, & 0, & 0, & 0, & 0\end{bmatrix} + \begin{bmatrix}-1, 0, 0, 0, 0, 0\end{bmatrix} \qquad = \begin{bmatrix}0, & 0, & 0, & 0, & 0, & 0\end{bmatrix}$$
$$g_1[n] = (\begin{bmatrix}1, & 0, & 0, & 0, & 0, & 0\end{bmatrix} - \begin{bmatrix}-1, 0, 0, 0, 0, 0\end{bmatrix}) \; W_{12}^n \quad = \begin{bmatrix}2, & 0, & 0, & 0, & 0, & 0\end{bmatrix}$$

$X(2k)$ are computed from $g_1[n]$, therefore, all samples are 0.

$X(2k+1)$ are computed from $g_2[n]$, where all samples are 0, except for $n = 0$. So,

$$X(2k+1) = 2\,W_6^0$$

In summary, $X[k] = [0, 2 , 0 , 2, 0 , 2 , 0 , 2 , 0 , 2 , 0 , 2]$.

(f) 1 pnt See Figure 3.

Question 2 (8 points)

Let us consider an analog signal that is stationary with zero mean and with a range between -2 and 2 mV. During analog-to-digital conversion, it is quantized with a uniform quantizer to 3 bits plus a sign bit.

(a) Compute the average power of the quantization noise. (Model the quantization noise as in the book, i.e. stationary white noise uncorrelated with the signal.)

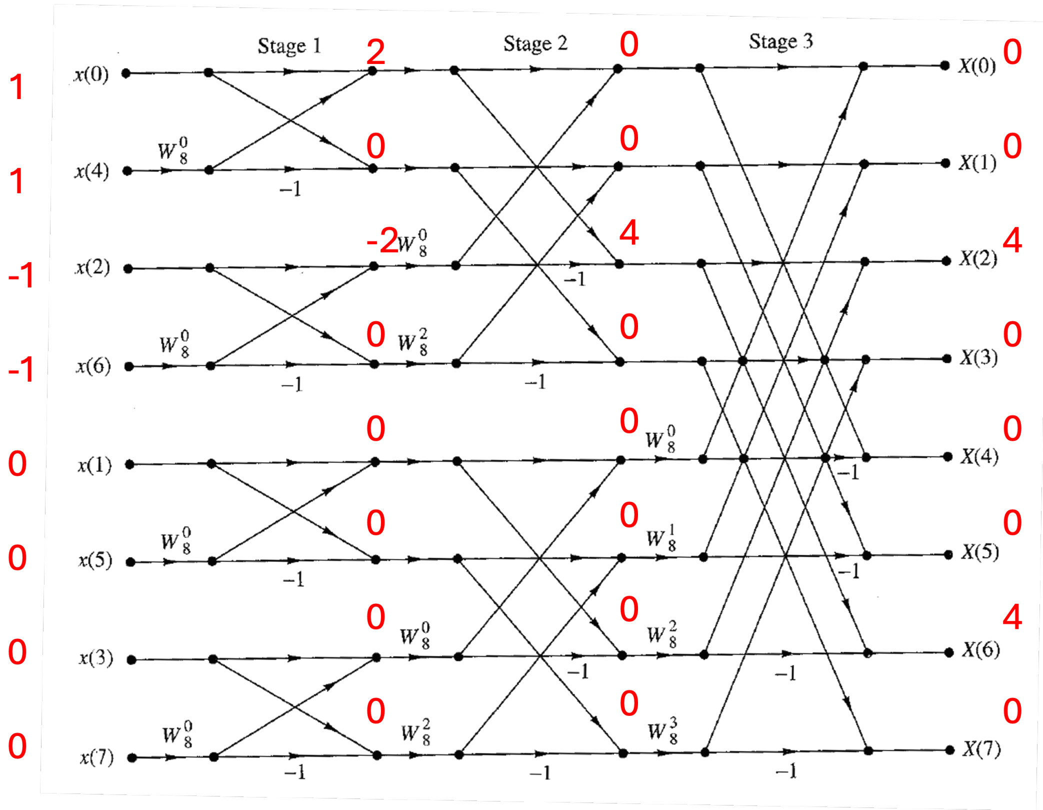
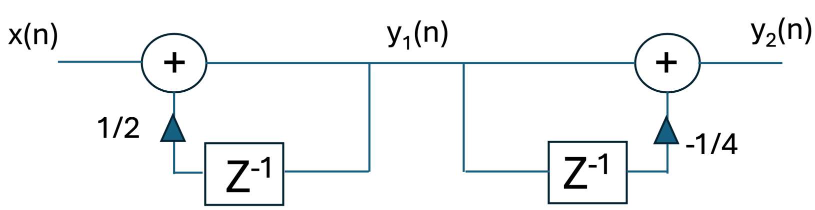


Figure 3.

(b) How can I increase the SQNR of the quantization process, while keeping the number of bits the same? Name at least 2 different strategies. (No need for lengthy explanation, just 2 keywords.)

Let us now consider the following digital system with 2 first-order filter sections, where the outputs of the multipliers are quantized in the same way as $x[n]$:



(c) Calculate the impulse response of the system.

(d) How much is the variance of the quantization noise at the output of the system, considering the errors introduced at the multipliers?

Solution

(a) 2 pnt $P_n = \sigma_z^2 = \dfrac{\Delta^2}{12}$, where $\Delta = \dfrac{R}{2^{b+1}} = \dfrac{4}{2^4}$. Therefore, $P_n = \frac{1}{12 \cdot 16} = 0.0052\,(\mathrm{mV})^2$.

(b) 1 pnt Using oversampling or predictive quantization.

(c) 2 pnt First, calculate the impulse response of the first section, which is a first order IIR filter, so $h_1[n] = (\frac{1}{2})^n u[n]$. The second section has the following system function:

$$h_2[n] = h_1[n] - \frac{1}{4} h_1[n-1]$$

Substituting the impulse response of the first section gives

$$h[n] = (\frac{1}{2})^n u[n] - \frac{1}{4}(\frac{1}{2})^{n-1} u[n-1] = \frac{1}{2}\delta[n] + (\frac{1}{2})^{n+1} u[n]$$

The answer can be written in several other forms, e.g.

$$h[n] = [\cdots, 0, \boxed{1}, \frac{1}{4}, \frac{1}{8}, \cdots]$$

or $h[n] = \delta[n] + (\frac{1}{2})^{n+1} u[n-1]$.

(d) 3 pnt The impulse response from the first noise source to the output is the same as the impulse response of the system itself, i.e. as above.

The second noise source is directly at the output of the system, so the impulse response is $\delta[n]$. Therefore,

$$\begin{aligned}
\sigma_d^2 &= \sigma_z^2 \left(\sum_{n=0}^{\infty} (h[n])^2 \ + 1^2 \right) \\
&= \sigma_z^2 \left(1^2 + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{2(n+1)} \ + 1^2 \right) \\
&= \sigma_z^2 \left(1^2 + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{2(n+1)} \ - \left(\frac{1}{2}\right)^{(0)} + 1^2 \right) \\
&= \sigma_z^2 \left(\frac{1}{4} \sum_{n=0}^{\infty} \frac{1}{4}^n \ + 1^2 \right) \\
&= \sigma_z^2 \left(\frac{1}{4} \frac{1}{1 - \frac{1}{4}} + -\frac{1}{4} 1 \right) \\
&= 0.0052 \cdot 2.083 \quad = \quad 0.0108
\end{aligned}$$

Question 3 (6 points)

Let M_n be a sequence of independent random numbers ("bits"), where $M_n \in \{0, 1\}$ with equal probabilities. Further let $p(t)$ be a pulse,

$$p(t) = \begin{cases} 1, & -0.5 \leq t \leq 0.5 \\ 0, & \text{otherwise} \end{cases}$$

and for $T = 1$ consider the random process

$$X(t) = \sum_{n=-\infty}^{\infty} M_n \, p(t - nT)$$

(a) Draw three different realizations of $X(t)$.

(b) What type of random process is $X(t)$? [Think of continuous value/discrete value; continuous-time/discrete time.]

(c) Compute the probability mass function (PMF) $P_{X(t)}(x)$. Is this a complete description of the random process?

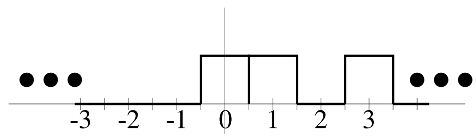
(d) Compute $\mathrm{E}[X(t)]$.

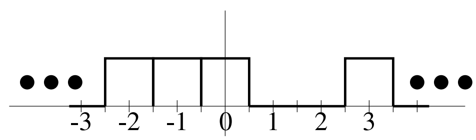
(e) Compute the autocorrelation function $R_X(t, \tau)$ for $t = 0$, i.e. compute $R_X(0, \tau)$.

(f) Is M_n a stationary random process? Is $X(t)$ stationary? Is it WSS?

Solution

(a) 1 pnt This is an ASK modulated communication signal. Examples are:





(b) 1 pnt This is a discrete value continuous-time random process. (Therefore, $X(t)$ is described by a PMF.)

(c) 1 pnt

$$P_{X(t)}(x) = \begin{cases} \frac{1}{2} & x = 0 \\ \frac{1}{2} & x = 1 \\ 0 & \text{otherwise} \end{cases}$$

This does not fully describe the random process, e.g., it doesn't say anything about the relations of 2 samples $X(t)$ and $X(t + \tau)$.

(d) 1 pnt $\mathrm{E}[X(t)] = \mathrm{E}[\sum_{n=-\infty}^{\infty} M_n\, p(t - nT)] = \sum_{n=-\infty}^{\infty} \mathrm{E}[M_n]\, p(t - nT) = \frac{1}{2}\sum_{n=-\infty}^{\infty} p(t - nT) = \frac{1}{2} \cdot 1 = \frac{1}{2}$.

(e) 1 pnt For $t = 0$ and $|\tau| < 0.5$, we have $X(0) = X(\tau)$, hence

$$R_X(0, \tau) = \mathrm{E}[X(0)X(\tau)] = \mathrm{E}[X(0)^2] = \frac{1}{2}$$

For $t = 0$ and $\tau > 0.5$, we have that $X(0)$ is independent of $X(\tau)$, hence $R_X(0, \tau) = 0 + \left(\frac{1}{2}\right)^2 = \frac{1}{4}$. Altogether,

$$R_X(0, \tau) = \begin{cases} \frac{1}{2} & |\tau| < 0.5 \\ \frac{1}{4} & \text{otherwise} \end{cases}$$

(f) 1 pnt M_n is a stationary random process because it is iid. (Incorrect arguments: M_n is not a function of n or t, or the PMF is not a function of n or t.)

$X(t)$ is not stationary because the joint probability $P_{X(t_1),X(t_2)}(x_1, x_2)$ is not always equal to a shifted version $P_{X(t_1+\tau),X(t_2+\tau)}(x_1, x_2)$. Another argument: $X(t)$ is not WSS (see below) so it is not stationary either.

E.g., if $t_1 = -0.25$ and $t_2 = 0.25$ then $X(t_1)$ and $X(t_2)$ are equal to each other, but if we shift by $\tau = 0.5$ and consider $X(t_1 + \tau)$ and $X(t_2 + \tau)$ then these two samples are independent of each other.

Not WSS because $R_X(t, \tau)$ depends on t, for the same reason as above. (Writing down the exact expression is not a pleasant exercise.)

Question 4 (9 points)

For this question you might want to make use of Table 3, included at the end of this exam.

$$X[n] \longrightarrow \boxed{h[n]} \longrightarrow Y[n]$$

Consider a WSS iid random process $X[n]$ with mean $\mu_X = 2$ and variance $\sigma_X^2 = 3$. We filter $X[n]$ with an FIR filter $h[n]$; the output sequence $Y[n]$ is given by

$$Y[n] = X[n] - 2X[n-1]$$

(a) Determine the autocorrelation sequence $R_X[k]$ of the input.

(b) Compute μ_Y, the mean of the output random process.

(c) Compute the crosscorrelation sequence $R_{XY}[k]$.

(d) Compute the autocorrelation sequence $R_Y[k]$ of the output.

(e) Compute the power spectral density $S_X(\phi)$ of the input.

(f) Compute the power spectral density $S_Y(\phi)$ of the output.

(g) Compute the average power of the output.

$$Y[n] \longrightarrow \boxed{g[n]} \longrightarrow W[n]$$

Next, $Y[n]$ is filtered by a first-order AR filter with transfer function

$$G(z) = \frac{1}{1 - a\,z^{-1}}, \qquad |a| < 1$$

resulting in the output $W[n]$.

If we take $a = 2$, then we recover $X[n]$, however, this filter is not stable. Therefore, we compromise and will try to recover a signal that only has the same autocorrelation as $X[n]$.

(h) Determine a stable filter such that the resulting autocorrelation sequence matches that of $X[n]$ up to a constant c, i.e., $R_W[k] = c\,R_X[k]$.

Solution

(a) 1 pnt

$$R_X[k] = \sigma_X^2 \delta[k] + \mu_X^2 = 3\delta[k] + 4$$

(b) 1 pnt First derive $h[n] = \delta[n] - 2\delta[n-1]$. Then

$$\mu_Y = \mu_X \sum h[n] = 2(1-2) = -2\,.$$

Alternatively, from $Y[n] = X[n] - 2X[n-1]$ it follows

$$\mathrm{E}[Y[n]] = \mathrm{E}[X[n]] - 2\mathrm{E}[X[n-1]] = \mu_X - 2\mu_X = -2.$$

(c) 1 pnt

$$R_{XY}[k] = h[k] * R_X[k] = (\delta[k] - 2\delta[k-1]) * (3\delta[k] + 4) = 3\delta[k] - 6\delta[k-1] - 4$$

where the last term is computed using $h[k] * 4 = \sum h[k] 4 = -4$.

Alternatively, show this using

$$R_{XY}[k] = h[k]*R_X[k] = R_X[k] - 2R_X[k-1] = 3\delta[k] + 4 - 2(3\delta[k-1]+4) = 3\delta[k] - 6\delta[k-1] - 4$$

(d) 1 pnt

$$R_Y[k] = h[-k]*R_{XY}[k] = (-2\delta[k+1]+\delta[k])*(3\delta[k]-6\delta[k-1]-4) = -6\delta[k+1]+15\delta[k]-6\delta[k-1]+4\,.$$

This could also be found by first calculating $h[k] * h[-k] = [\cdots 0, -2, \boxed{5}, -2, 0 \cdots]$, and then applying this to $R_X[k]$:

$$R_Y[k] = -2R_X[k+1] + 5R_X[k] - 2R_X[k-1] = \cdots = -6\delta[k+1] + 15\delta[k] - 6\delta[k-1] + 4\,.$$

(e) 1 pnt Take the DTFT of $R_X[k]$: using Table 3,

$$S_X(\phi) = 3 + 4\delta(\phi)$$

(f) 1 pnt Take the DTFT of $R_Y[k]$:

$$S_Y(\phi) = 15 - 6(e^{-j2\pi\phi} + e^{-j2\pi\phi}) + 4\delta(\phi) = 15 - 12\cos(2\pi\phi) + 4\delta(\phi)$$

(g) 1 pnt The average output power is

$$R_Y[0] = 15 + 4 = 19$$

This would also follow by integrating $S_Y(\phi)$ over the interval $-\pi/2 \le \phi \le \pi/2$.

(h) 2 pnt Note that

$$\begin{aligned} |H(e^{j2\pi\phi})|^2 &= (1 - 2\,e^{-j2\pi\phi})(1 - 2\,e^{j2\pi\phi}) = 5 - 4\cos(2\pi\phi) \\ |G(e^{j2\pi\phi})|^2 &= \frac{1}{1 - a\,e^{-j2\pi\phi}} \frac{1}{1 - a\,e^{j2\pi\phi}} = \frac{1}{1 + a^2 - 2a\cos(2\pi\phi)} \end{aligned}$$

Since $S_W(\phi) = |G(e^{j2\pi\phi})|^2 S_Y(\phi) = |G(e^{j2\pi\phi})|^2 |H(e^{j2\pi\phi})|^2 S_X(\phi) = c\,S_X(\phi)$, we need

$$\frac{5 - 4\cos(2\pi\phi)}{1 + a^2 - 2a\cos(2\pi\phi)} = c$$

To obtain a constant (i.e., independent of frequency ϕ), we need to set

$$\frac{4}{5} = \frac{2a}{1 + a^2}$$

Solving for a gives $a = \frac{1}{2}$ or $a = 2$. The stable solution is $a = \frac{1}{2}$.