Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Signal Processing Systems

Resit exam EE2S31 SIGNAL PROCESSING
17 July 2024 13:30–16:30

Closed book; two sides of one A4 with handwritten notes permitted. No other tools except a basic pocket calculator permitted. Note the attached tables!

This exam consists of five questions (30 points). Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.

Question 1 (7 points)

Let $x_a(t)$ be an analog signal with center frequency 25 kHz and bandwidth 10 kHz. Suppose the signal $x_a(t)$ is uniformly sampled at F_s samples/sec to obtain a discrete-time signal $x[n]$.

(a) What is the Nyquist sampling rate for $x_a(t)$?

(b) What is the minimum sampling rate F_s required to reconstruct the signal without any distortion?

Let the sampling rate $F_s = 40$ kHz. We collect N samples, and want to use a DFT of $\hat{x} = [x[0], x[1], \dots, x[N-1]]$ to estimate the spectrum of $x_a(t)$ with a spectral resolution less than or equal to 100 Hz, by suitably choosing N.

(c) To achieve this, what is the minimum duration T in seconds of the analog signal recorded? What is the corresponding value of N?

Suppose we increase the number of frequency bins in the DFT without increasing the number of samples. Consider the following $2N$-point DFTs,

$$\hat{X}_1[k] = \text{DFT}\left([x[0], x[1], \dots, x[N-1], \underbrace{0, 0, \dots, 0}_{N \text{ zeros}}]\right)$$

$$\hat{X}_2[k] = \text{DFT}\left([\underbrace{0, 0, \dots, 0}_{N \text{ zeros}}, x[0], x[1], \dots, x[N-1]]\right)$$

$$\hat{X}_3[k] = \text{DFT}\left([x[0], x[1], \dots, x[N-1], x[0], x[1], \dots, x[N-1]]\right).$$

(d) Which of the above options (if any) can increase the spectral resolution?

(e) What is the relation between $\hat{X}_1[k]$ and $\hat{X}_2[k]$? (Express $\hat{X}_2[k]$ in terms of $\hat{X}_1[k]$.)

(f) What is the relation between $\hat{X}_3[k]$ and $\hat{X}_1[k]$? (Express $\hat{X}_3[k]$ in terms of $\hat{X}_1[k]$.)

Consider a filter $h = [h[0], h[1], \dots, h[N-1]]$. Let the linear convolution of $\hat{x}$ and h be denoted by $y[n]$. Also, $z[n]$ denotes the N-point inverse DFT of the product of the N-point DFTs of $\hat{x}$ and h.

(g) What are the values of n for which $y[n] = z[n]$?

Solution

1p (a) The signal occupies frequencies between 20 kHz and 30 kHz. So, the Nyquist sampling rate is $2 \times 30 = 60$ kHz.

1p (b) The minimum sampling rate is $2 \times 10 = 20$ kHz.

1p (c) The DFT resolution is $100 = F_s/N$, and the signal length $T = N/F_s = 1/100 = 10$ ms. Also, $N = 400$.

1p (d) None of the options can improve the spectral resolution.

1p (e) Using the circular shift property of DFT, we derive

$$\hat{X}_2[k] = e^{-j2\pi kN/2N}\hat{X}_1[k] = (-1)^k \hat{X}_1[k], \quad k = 0, 1, \dots, 2N-1 = 799$$

1p (f) For $k = 0, 1, \dots, 2N-1 = 799$

$$\hat{X}_3[k] = \hat{X}_1[k] + \hat{X}_2[k] = \begin{cases} 2\hat{X}_1[k] & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

1p (g) If $y[n] = z[n]$, we have

$$\sum_{m=0}^{N-1} x[m]h[n-m] = \sum_{m=0}^{N-1} x[m]h[(n-m)]_N.$$

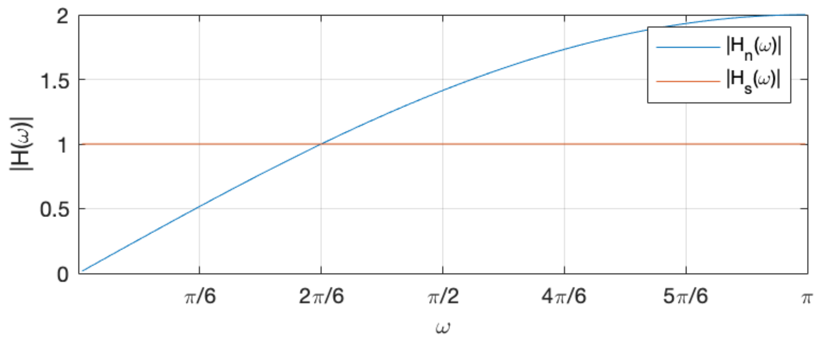
The above relation holds only if, for $m = 0, 1, \dots, N-1$,

$$n - m = [n-m]_N = \begin{cases} N + (n-m) & \text{if } n < m \\ n - m & \text{if } n \geq m. \end{cases}$$

Hence, $y[n] = z[n]$ if $n \geq m$ for $m = 0, 1, \dots, N-1$, implying $n = N - 1 = 399$.

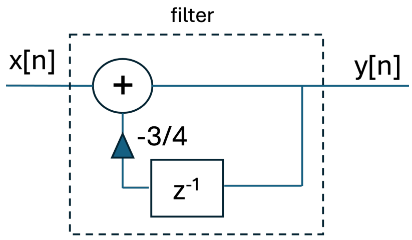
Question 2 (7 points)

An analog signal $x_a(t)$ with bandwidth B is sampled using a first order sigma-delta modulator (SDM) at a sampling rate $F_s = 12$ kHz. The noise and signal transfer function of the SDM ($H_n(\omega)$ and $H_s(\omega)$, respectively) of the SDM are as shown in the next figure:



(a) What is the maximum bandwidth B, such that the SDM at the given sampling rate suppresses the noise in the signal band?

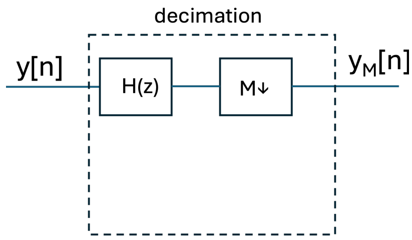
A digital signal $x[n]$ is filtered to obtain $y[n]$ as shown in the next figure:



In the implementation of the filter, the output of the multiplier is quantized to 3 bits plus a sign bit using sign-magnitude representation. The range of the quantizer is $[-1,\ 1]$. The result is rounded downward in case it is equal to the decision threshold.

(b) Consider $x[n] = \frac{7}{8}\delta[n]$. What are the first three quantized output samples of $y[n]$? Write down the values and their binary representation as well.

A digital signal $y[n]$ is decimated by a factor $M = 2$ to a sampling rate of 6 kHz as shown next:



(c) What is the role of $H(z)$ in the decimator?

To make the decimation system more efficient, I want to switch the order of the downsampler and the filter $H(z)$. For this, I will first use a polyphase representation of the $H(z)$. Let us assume that the system function is $H(z) = 1 + 0.4z^{-1} - 0.1z^{-2} + 0.2z^{-3}$.

(d) Write down the system functions of the polyphase components $P_i(z)$.

(e) First, draw the block diagram of the system after applying the polyphase representation. Then, draw the diagram of the new, more efficient system (i.e., after switching the order).

(f) In what sense is the new system more efficient?

Solution

1p (a) The noise is suppressed below frequencies $2\pi/6$. Thus, B should not exceed $F_s/6 = 2$ kHz.

2p (b)

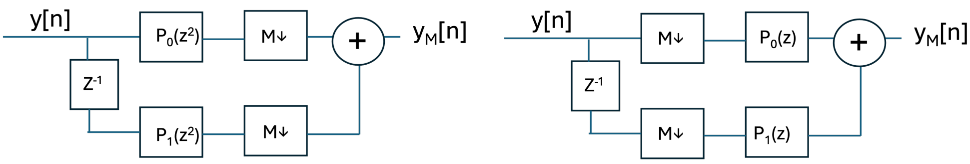
$$
\begin{aligned}
& y[n] = x[n] + \mathcal{Q}[-\frac{3}{4}y[n-1]] \\
\Rightarrow\quad & y[0] = x[0] + \mathcal{Q}[-\frac{3}{4}y[-1]] = \frac{7}{8} = (0.111)_2 \\
& y[1] = x[1] + \mathcal{Q}[-\frac{3}{4}y[0]] = 0 + \mathcal{Q}[-\frac{3}{4}\cdot\frac{7}{8}] = \mathcal{Q}[-\frac{21}{32}] = -\frac{5}{8} = (1.101)_2 \\
& y[2] = x[2] + \mathcal{Q}[-\frac{3}{4}y[1]] = 0 + \mathcal{Q}[\frac{3}{4}\cdot\frac{5}{8}] = \mathcal{Q}[\frac{15}{32}] = \frac{4}{8} = (0.100)_2
\end{aligned}
$$

1p (c) It is a low-pass filter that eliminates the spectrum of the signal above π/M.

1p (d) We need a 2-component polyphase representation. Thus $H(z) = P_0(z^2) + z^{-1}P_1(z^2)$, with

$$P_0(z) = 1 - 0.1z^{-1}$$
$$P_1(z) = 0.4 + 0.2z^{-1}$$

1p (e))



The figure shows the decimation system after applying the polyphase representation (left) and after switching the order of the filters and the downsampler (right).

Note: On the left figure the downsampling can also take place after the adder.

1p (f) Now the decimation filters run at half the original sampling rate.

Question 3 (5 points)

The joint probability density function of two random variables X and Y is given by

$$f_{X,Y}(X,Y) = \begin{cases} c & \text{for } 0 \leq y \leq \frac{1}{2}x, \quad 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Calculate the value of the constant c.

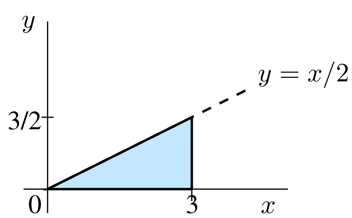
(b) Derive the marginal pdfs $f_X(x)$ and $f_Y(y)$.

(c) Argue whether the random variables X and Y are independent or not.

(d) Determine the conditional probability $f_{X|Y}(x|y)$.

(e) Determine the MMSE estimator of X if we have observed $Y = y$.

Solution



1p (a)

$$\int_0^{3/2}\int_{2y}^{3} c\,\mathrm{d}x\mathrm{d}y = \int_0^{3/2}\left[c\,x\right]_{2y}^{3}\mathrm{d}y$$
$$= \int_0^{3/2} c(3-2y)\mathrm{d}y$$
$$= c\left[3y-y^2\right]_0^{3/2}$$
$$= c\left(\frac{9}{2}-\frac{9}{4}\right) = \frac{9}{4}c \quad = 1$$

or

$$\int_0^{3}\int_0^{x/2} c\,\mathrm{d}y\mathrm{d}x = \int_0^{3}\left[c\,y\right]_0^{x/2}\mathrm{d}x$$
$$= \int_0^{3}\frac{c}{2}\,x\,\mathrm{d}x$$
$$= \left[\frac{c}{4}x^2\right]_0^{3}$$
$$= \frac{9}{4}c \quad = 1$$

Hence, $c = 4/9$.

1p (b) Within the ranges of X and Y, we find

$$f_X(x) = \int_{y=0}^{x/2} c\,\mathrm{d}y = \frac{c}{2}x\,, \qquad f_Y(y) = \int_{x=2y}^{3} c\,\mathrm{d}x = 3c - 2cy\,.$$

Altogether, this gives

$$f_X(x) = \begin{cases} \frac{1}{2}cx & \text{for } 0 \le x \le 3 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} 3c-2cy & \text{for } 0 \le y \le \frac{3}{2} \\ 0 & \text{otherwise} \end{cases}$$

1p (c) X and Y are not independent, because $f_X(x)\,f_Y(y) \neq f_{X,Y}(x,y)$.

1p (d)

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \dfrac{1}{3-2y} & \text{for } 2y \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

1p (e)

$$\hat{X}(y) = \mathrm{E}[X|Y=y] = \int_{2y}^{3}\frac{x}{3-2y}\mathrm{d}x = \left[\frac{x^2}{6-4y}\right]_{2y}^{3} = \frac{9-4y^2}{6-4y} = \frac{3+2y}{2}\,.$$

Question 4 (5 points)

Let $X_1 \sim \mathrm{Exp}(\lambda_1)$ and $X_2 \sim \mathrm{Exp}(\lambda_2)$ be two independent exponentially distributed random variables, with known parameters λ_1 and λ_2. *See table.*

The random variable Z is defined as $Z = X_1 + X_2$.

(a) Assuming $\lambda_1 = 2$ and $\lambda_2 = 3$, use the Chebyshev inequality to find a bound on $\mathrm{P}[Z > 3]$.

(b) Show that the moment generating function $\phi_Z(s)$ of Z can be written as

$$\phi_Z(s) = \frac{A}{\lambda_1 - s} + \frac{B}{\lambda_2 - s}$$

and determine A and B as function of λ_1 and λ_2.

(c) Calculate the probability $\mathrm{P}[Z > 3]$.

Hint: first calculate the pdf $f_Z(z)$. Keep the answer as function of λ_1 and λ_2.

Solution

2p (a) First determine $\mathrm{E}[Z] = \mathrm{E}[X_1] + \mathrm{E}[X_2] = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ and $\mathrm{var}[Z] = \mathrm{var}[X_1] + \mathrm{var}[X_2] = \frac{1}{4} + \frac{1}{9} = \frac{13}{36}$.

$$\begin{aligned} \mathrm{P}[Z \geq 3] &= \mathrm{P}\big[Z - \mathrm{E}[Z] \geq 3 - \mathrm{E}[Z]\big] \\ &\leq \mathrm{P}\big[|Z - \mathrm{E}[Z]| \geq 3 - \mathrm{E}[Z]\big] \\ &= \mathrm{P}\big[|Z - \mathrm{E}[Z]| \geq \frac{13}{6}\big] \\ &\leq \frac{\mathrm{var}[Z]}{(13/6)^2} \quad = \frac{1}{13} \approx 0.0769\,. \end{aligned}$$

1p (b) Since X_1 and X_2 are independent, $\phi_Z(z)$ is given by the product of the MGFs of X_1 and X_2 (see table):

$$\phi_Z(z) = \frac{\lambda_1}{\lambda_1 - s}\frac{\lambda_2}{\lambda_2 - s} = \frac{A}{\lambda_1 - s} + \frac{B}{\lambda_2 - s}$$

(using a partial fraction expansion) where we find

$$A = \frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}\,, \qquad B = -\frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}$$

which gives

$$\phi_Z(s) = \frac{\lambda_2}{\lambda_2 - \lambda_1}\frac{\lambda_1}{\lambda_1 - s} + \frac{\lambda_1}{\lambda_1 - \lambda_2}\frac{\lambda_2}{\lambda_2 - s} = \frac{6}{2 - s} + \frac{6}{3 - s}\,.$$

2p (c) Use the inverse Laplace transform (table) to find

$$f_Z(z) = \frac{\lambda_2}{\lambda_2 - \lambda_1}\lambda_1 e^{-\lambda_1 z} + \frac{\lambda_1}{\lambda_1 - \lambda_2}\lambda_2 e^{-\lambda_2 z} = \frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}\big(e^{-\lambda_1 z} - e^{-\lambda_2 z}\big)$$

(for $z \geq 0$, and 0 otherwise). Next,

$$\begin{aligned} \mathrm{P}[Z > 3] &= \int_3^\infty f_Z(z)\,\mathrm{d}z \\ &= \int_3^\infty \frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}\big(e^{-\lambda_1 z} - e^{-\lambda_2 z}\big)\mathrm{d}z \\ &= \frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}\left(\frac{1}{\lambda_1}e^{-3\lambda_1} - \frac{1}{\lambda_2}e^{-3\lambda_2}\right) \approx 0.00719\,. \end{aligned}$$

Question 5 (6 points)

Consider a linear time-invariant (LTI) system with as input the random process $X(t)$, and as output the random process $Y(t)$. The impulse response $h(t)$ of the system is given by

$$h(t) = \begin{cases} \frac{1}{T} & \text{for } 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

First, consider that $X(t)$ is uncorrelated, zero mean, with variance σ_X^2.

(a) Determine the output power spectral density $S_Y(f)$.

Now, consider that $X(t)$ is given by

$$X(t) = \begin{cases} A\,t & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where A is a continuous random variable that is uniformly distributed between 0 and 1.

(b) Plot three realizations of $X(t)$ and argue whether or not this process is (1) stationary, (2) ergodic.

(c) What type of random process is $X(t)$? [Think of continuous value/discrete value; continuous-time/discrete time.]

(d) Calculate the autocorrelation function $R_X(t, \tau)$.

(e) Calculate $\mathrm{E}[X(t)]$ and $\mathrm{E}[Y(t)]$.

Hint: Distinguish in the calculation the cases $t \leq 0$, $0 \leq t \leq T$, and $t \geq T$.

Solution

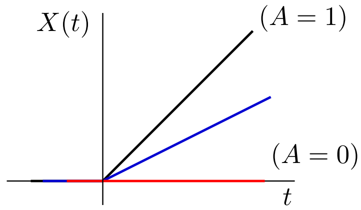
1p (a) $S_Y(f) = |H(f)|^2\, S_X(f)$, with $S_X(f) = \sigma_X^2$ and

$$h(t) = \frac{1}{T}\mathrm{rect}\left(\frac{t - T/2}{T}\right) \quad \Rightarrow \quad H(f) = \mathrm{sinc}(fT)e^{-j\pi fT}\,.$$

Hence

$$S_Y(f) = |\mathrm{sinc}(fT)|^2\, \sigma_X^2\,.$$

1p (b)



This process is not stationary as the expected value of $X(t)$ is time dependent. Moreover, due to the time dependency, time averages are not the same as ensemble averages, and therefore this process is also not ergodic.

1p (c) Continuous-time, continuous-value.

1p (d) We have $\mathrm{E}[A^2] = \left(\frac{1}{2}\right)^2 + \frac{1}{12} = \frac{1}{3}$, and, for $t \geq 0, t + \tau \geq 0$:

$$R_X(t, \tau) = \mathrm{E}[X(t)X(t + \tau)] = \mathrm{E}[A^2](t^2 + t\tau) = \frac{1}{3}(t^2 + t\tau)$$

so that

$$R_X(t, \tau) = \begin{cases} \frac{1}{3}(t^2 + t\tau) & t \geq 0, t + \tau \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

2p (e) We have $\mathrm{E}[A] = \frac{1}{2}$, and

$$\mathrm{E}[X(t)] = \begin{cases} \mathrm{E}[A]t & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{2}t & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
\mathrm{E}[Y(t)] &= \int_{-\infty}^{\infty} h(\tau)\mathrm{E}[X(t-\tau)]\mathrm{d}\tau = \begin{cases} 0 & \text{for } t < 0 \\ \dfrac{1}{2T}\displaystyle\int_0^t (t-\tau)\mathrm{d}\tau & \text{for } 0 \leq t \leq T \\ \dfrac{1}{2T}\displaystyle\int_0^T (t-\tau)\mathrm{d}\tau & \text{for } t \geq T \end{cases} \\
&= \begin{cases} 0 & \text{for } t < 0 \\ \dfrac{1}{2T}\left[t\tau - \dfrac{1}{2}\tau^2\right]_0^t & \text{for } 0 \leq t \leq T \\ \dfrac{1}{2T}\left[t\tau - \dfrac{1}{2}\tau^2\right]_0^T & \text{for } t \geq T \end{cases} \\
&= \begin{cases} 0 & \text{for } t < 0 \\ \dfrac{t^2}{4T} & \text{for } 0 \leq t \leq T \\ \dfrac{t}{2} - \dfrac{T}{4} & \text{for } t \geq T \end{cases}
\end{aligned}$$

Note: $X(t)$ is not WSS so $\mathrm{E}[Y(t)] \neq \mu_X \int h(\tau) d\tau$.