Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Signal Processing Systems

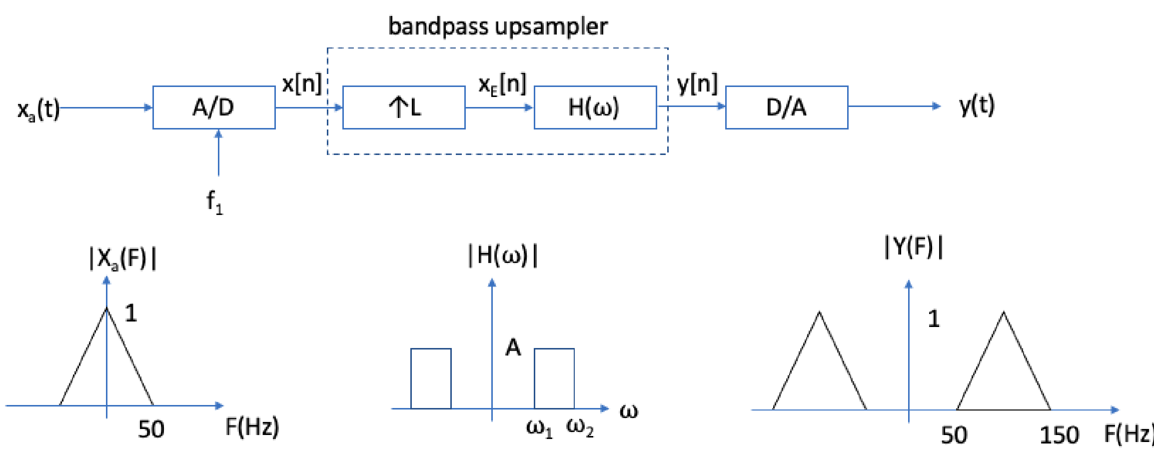
Partial exam EE2S31 SIGNAL PROCESSING
Part 2: 30 June 2023 (9:00-11:00)

Closed book; two sides of one A4 with handwritten notes permitted. No other tools except a basic pocket calculator permitted. Note the attached tables!

This exam consists of four questions (35 points). Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.
Hint: Avoid losing too much time on detailed calculations, write down the general approach first.

Question 1 (10 points)

The discrete-time filtering system shown below comprises an A/D converter sampling at rate $f_1 = 100$ Hz, an upsampler with $L = 4$, a bandpass filter with cut-off frequencies ω_1 and ω_2 and gain A, and a D/A converter. The spectra of the input ($X_a(F)$) and the output ($Y_a(F)$) of the system are also shown. The upsampler and the bandpass filter can be considered together as a 'bandpass upsampler'.



(a) Sketch the spectra of $x[n]$ and $x_E[n]$! Make sure that you indicate the magnitudes correctly, and indicate both normalized angular frequencies (ω) and physical frequencies (F) on your graphs!

(b) If $x[n] = \{1, 2, 3, 2, 1\}$, what is the sequence $x_E[n]$?

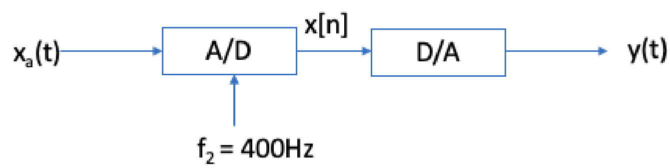
(c) What are the cut-off normalized angular frequencies ω_1 and ω_2 of the filter $H(\omega)$, considering the spectrum of $x_E[n]$ and the given output spectrum $Y(F)$?

(d) The transfer function of $H(\omega)$ can be written in the form of a two-component polyphase decomposition as

$$H(z) = P_0(z^2) + z^{-1}P_1(z^2)$$

Use this polyphase decomposition to design a more efficient 'bandpass upsampler' and sketch the new filter structure!

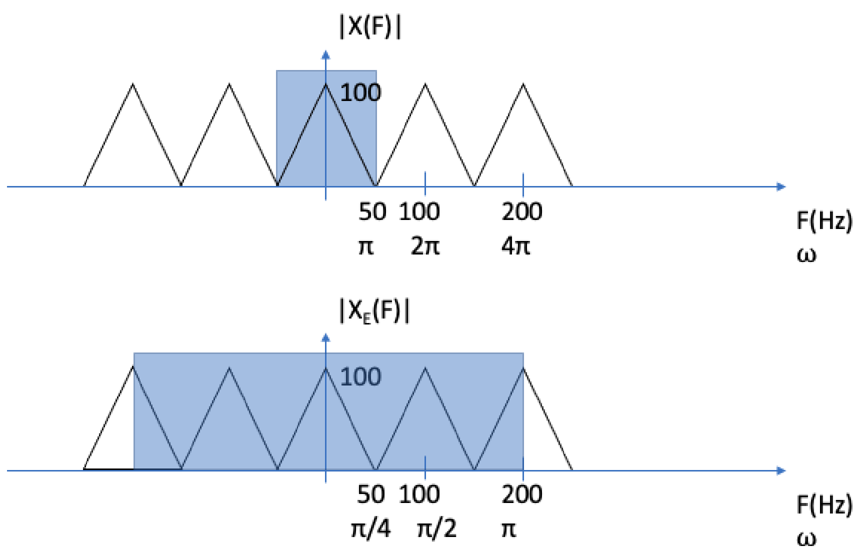
(e) Returning to the original system, let us redesign $H(\omega)$: our purpose is now not anymore to bandpass filter the signal. Instead, we want design a new $H(\omega)$ in order to get a system that produces an equivalent output as the alternative system shown below. Give the necessary filter specifications!



(f) Which system is better in terms of signal to quantization noise ratio: the original system (with the new $H(\omega)$), or the alternative system? Explain!

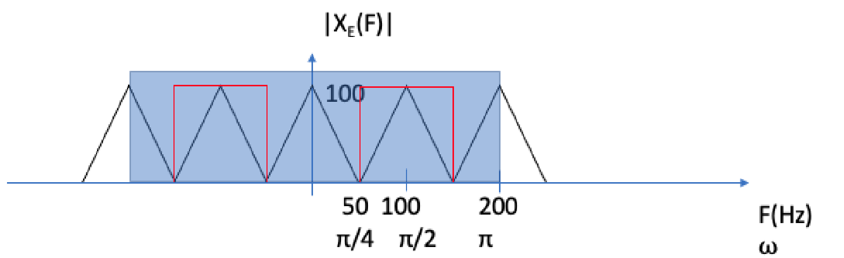
Solution

(a) 2 pnt The digital spectrum is periodic with 2π, i.e. 100 Hz. The upsampled spectrum is compressed relative to the normalized frequency axis, now 2π, is equivalent to 400 Hz. Therefore, three extra copies of the spectrum are within the fundamental frequency interval (shaded area).

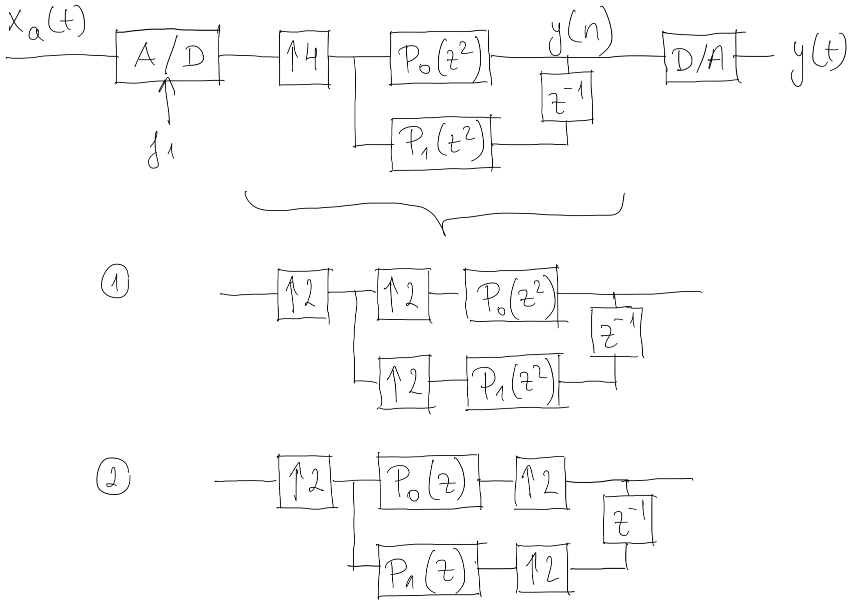


(b) 1 pnt $x_E[n] = \{1, 0, 0, 0, 2, 0, 0, 0, 3, 0, 0, 0, 2, 0, 0, 0, 1, 0, 0, 0\}$.

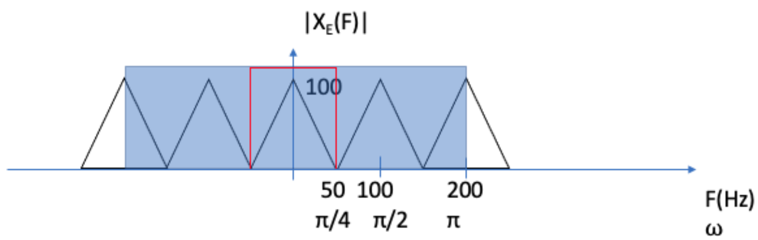
(c) 1 pnt To obtain $|Y(F)|$ at the output, a digital filter indicated in the spectrum in red is needed. The cut-off frequencies are $\omega_1 = \pi/4$ and $\omega_1 = 3\pi/4$.



(d) 2 pnt See figure below:



(e) 2 pnt It has to be a low-pass filter that removes the extra copies from the fundamental frequency band, as shown below, with a gain of 4 (note that the sampled spectrum produced by the alternative system has a maximum amplitude of 400 due to the higher sampling frequency!)



(f) 2 pnt The second system is better: the quantization noise power is independent of the sampling rate and is therefore equal in both systems. However, the signal power is higher in the second system, as the sampled spectrum has 4x higher magnitude due to the 4x higher sampling rate. Remember: $X(F) = F_s \sum_{k=-\infty}^{\infty} X_a(F - k \cdot F_s)$.

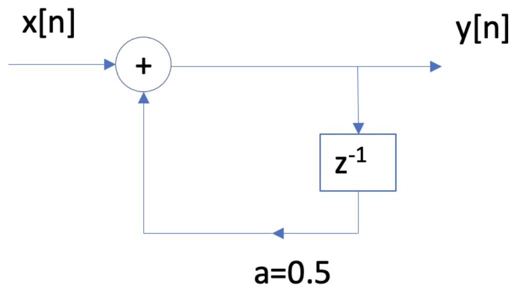
Question 2 (7 points)

Let us consider the quantization and coding of analog signals in the range between -1 and 1.

(a) Is quantization a linear process? Is it an invertible process? Explain your answer!

(b) Given N bits plus a sign bit, and assuming we use a simple sign-magnitude representation, what is the largest quantization level? What is the corresponding code word?

Let us now consider the following first-order system:



The quantization that takes place after multiplication uses $N = 3$ bits and rounds up (i.e., if the value to be quantized is equal to the decision threshold, and therefore it is at equal distance from 2 quantization levels, the higher quantization level is assigned). Due to the quantization, the system does not produce the ideal output $y[n]$; instead, it produces $v[n]$ as

$$v[n] = Q\{av[n-1]\} + x[n]$$

where $Q\{\cdot\}$ denotes the quantization operator.

(c) With an input $x[n] = \frac{3}{8}\delta[n]$, write down the first 5 output values ($v[n]$) of the system!

(d) Compare this output to the output of the ideal infinite-impulse response system. What is the difference? Explain! (no need to calculate)

Solution

(a) 1 pnt The function that maps input values to quantization level is non-linear (it is piece-wise linear) and not invertible.

(b) 2 pnt The largest quantization value is $1 - 2^{-N}$, and the corresponding codeword is 0.1...1 (N 1s after the sign-but which is 0).

(c) 3 pnt

$$v[0] = Q\{av[-1]\} + x[0] = Q\{\frac{1}{2} \cdot 0\} + \frac{3}{8} = \frac{3}{8}$$
$$v[1] = Q\{av[0]\} + x[1] = Q\{\frac{1}{2} \cdot \frac{3}{8}\} + 0 = Q\{\frac{3}{16}\} = \frac{2}{8}$$

Similarly,

$$v[2] = \frac{1}{8}$$
$$v[3] = \frac{1}{8}$$
$$v[4] = \frac{1}{8}$$

(d) 1 pnt The ideal system's output converges to 0 at infinity, while the quantized system reaches a limit cycle.

Question 3 (9 points)

Let A be a Uniform$(1, 2)$ random variable, and consider the random process

$$X(t) = A + \cos(2\pi t)\,.$$

(a) Draw 3 realizations of $X(t)$.

(b) Is the random process $X(t)$ discrete-time or continuous-time; discrete-value or continuous-value?

(c) Find the CDF $F_{X(t)}(x)$ and the PDF $f_{X(t)}(x)$.

(d) Is $f_{X(t)}(x)$ a complete description of the statistics of $X(t)$? (Motivate.)

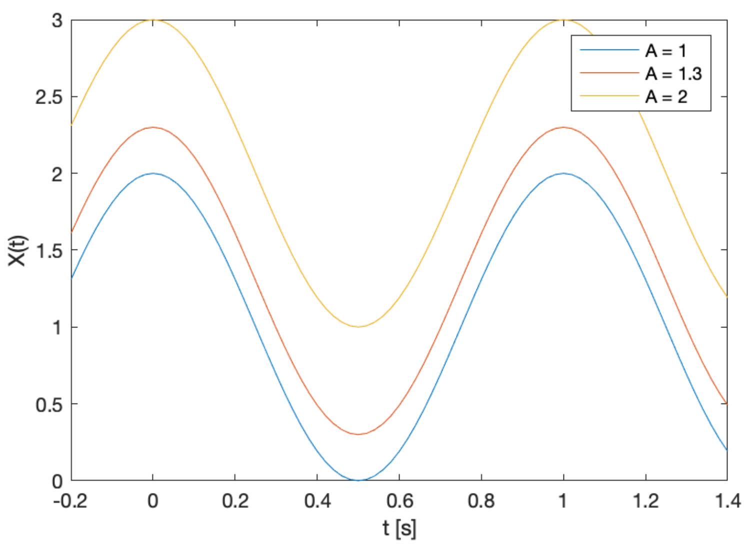
(e) Find the mean $\mathrm{E}[X(t)]$.

(f) Find the autocovariance $C_X(t, \tau)$ and the autocorrelation $R_X(t, \tau)$.

(g) Is $X(t)$ stationary? WSS? Ergodic? (Motivate.)

Solution

(a) 1 pnt



(b) 1 pnt Continuous-time, continuous-value

(c) 2 pnt Since A is Uniform$(1, 2)$, we have

$$f_A(a) = \begin{cases} 1 & 1 \le a \le 2 \\ 0 & \text{otherwise} \end{cases} \qquad\qquad F_A(a) = \begin{cases} 0 & a < 1 \\ a - 1 & 1 \le a \le 2 \\ 1 & a > 2 \end{cases}$$

Using this, we find

$$\begin{aligned}
F_{X(t)}(x) = \mathrm{P}[X(t) < x] &= \mathrm{P}[A + \cos(2\pi t) < x] \\
&= \mathrm{P}[A < x - \cos(2\pi t)] \\
&= \begin{cases} 0 & x - \cos(2\pi t) < 1 \\ x - \cos(2\pi t) - 1 & 1 \le x - \cos(2\pi t) \le 2 \\ 1 & x - \cos(2\pi t) > 2 \end{cases} \\
&= \begin{cases} 0 & x < 1 + \cos(2\pi t) \\ x - \cos(2\pi t) - 1 & 1 + \cos(2\pi t) \le x \le 2 + \cos(2\pi t) \\ 1 & x > 2 + \cos(2\pi t) \end{cases}
\end{aligned}$$

Taking the derivative, we find

$$f_{X(t)}(x) = \frac{\mathrm{d}}{\mathrm{d}x} F_{X(t)}(x) = \begin{cases} 1 & 1 + \cos(2\pi t) \le x \le 2 + \cos(2\pi t) \\ 0 & \text{otherwise} \end{cases}$$

(d) 1 pnt No, e.g., it does not say anything on the joint PDF of two samples, $f_{X(t_1),X(t_2)}(x_1, x_2)$, or the joint PDF of more than two samples.

(e) 1 pnt

$$\mu_X(t) = \mathrm{E}[X(t)] = \mathrm{E}[A] + \cos(2\pi t) = \frac{3}{2} + \cos(2\pi t)$$

(f) 2 pnt Since $X(t) - \mu_X(t) = A - \frac{3}{2}$,

$$C_X(t, \tau) = \mathrm{var}(A) = \frac{1}{12}$$

$$R_X(t, \tau) = C_X(t, \tau) + \mu_X(t)\mu_X(t + \tau) = \frac{1}{12} + \left(\frac{3}{2} + \cos(2\pi t)\right)\left(\frac{3}{2} + \cos(2\pi(t + \tau)\right) = \cdots$$

(g) 1 pnt Not stationary since $f_{X(t)}(x)$ depends on t (necessary condition not satisfied).

Not WSS since $\mu_X(t)$ depends on t.

Not ergodic since $X(t)$ is not stationary (necessary condition not satisfied).

Question 4 (9 points)

Let $X(t)$ be a WSS random process with mean $\mu_X = 2$ and autocovariance function $C_X(\tau) = \delta(\tau)$. We filter $X(t)$ with a lowpass filter with impulse response

$$h(t) = \begin{cases} e^{-t} & t \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

The output is $Y(t) = h(t) * X(t)$.

(a) Compute μ_Y, the mean of the output random process.

(b) Compute the crosscorrelation function $R_{XY}(\tau)$.

(c) Show that the autocorrelation function $R_Y(\tau)$ of the output has the form

$$R_Y(\tau) = a\, e^{-b|\tau|} + c$$

and determine the coefficients a, b and c.

(d) Compute the cross-power spectral density $S_{XY}(f)$.

(e) Compute the output power spectral density $S_Y(f)$; also make a plot of $S_Y(f)$ (carefully mark the values on the axes).

(f) What is the average output power?

Now, consider that the output of the filter is also disturbed by noise: let $Y(t) = h(t) {*} X(t) + W(t)$, where $W(t)$ is a white Gaussian noise process, independent of $X(t)$, with power spectral density $S_W(f) = 3$ W/Hz.

(g) Compute the autocorrelation $R_Y(\tau)$ of the output, and compute the output power spectral density $S_Y(f)$.

Solution

(a) 1 pnt

$$\mu_Y = \mu_X \int h(t)\mathrm{d}t = 2 \cdot \int_0^\infty e^{-t}\mathrm{d}t = 2$$

(b) 1.5 pnt $R_X(\tau) = 4 + \delta(\tau)$. Then

$$R_{XY}(\tau) = h(\tau) * R_X(\tau) = h(\tau) * (4 + \delta(\tau)) = 4 + h(\tau)$$

since $h(\tau) * 4 = \int 4h(t-\tau)\mathrm{d}t = 4\int h(t-\tau)\mathrm{d}t = 4$.

(c) 2 pnt

$$R_Y(\tau) = h(\tau) * h(-\tau) * R_X(\tau) = (4 + h(\tau)) * h(-\tau)$$

As above, $4 * h(-\tau) = 4$, and

$$h(\tau) * h(-\tau) = \int h(t)h(t+\tau)\mathrm{d}t = \int e^{-t}u(t)e^{-(t+\tau)}u(t+\tau)\mathrm{d}t$$

For $\tau > 0$:

$$\cdots = \int_0^\infty e^{-t}e^{-(t+\tau)}\mathrm{d}t = e^{-\tau}\int_0^\infty e^{-2t}\mathrm{d}t = \frac{1}{2}e^{-\tau}$$

For $\tau < 0$ we can do a similar calculation, or use the symmetry of $R_Y(\tau)$ to directly find

$$h(\tau) * h(-\tau) = \frac{1}{2}e^{-|\tau|}$$

Altogether,

$$R_Y(\tau) = \frac{1}{2}e^{-|\tau|} + 4$$

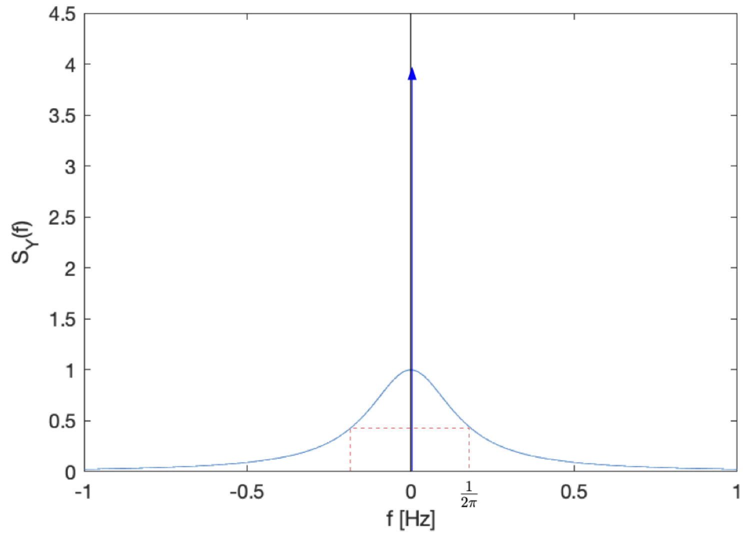
Alternatively (and probably easier) this can be computed in the frequency domain, using the results of the next two subquestions.

(d) 1 pnt

$$S_{XY}(f) = \mathcal{F}\{R_{XY}(\tau)\} = \frac{1}{1 + j2\pi f} + 4\delta(f)$$

(e) 1.5 pnt

$$S_Y(f) = \mathcal{F}\{R_X(\tau)\} = \frac{1}{1+(2\pi f)^2} + 4\delta(f)$$



(Note: the delta spike should be drawn separately as an arrow, indicating it goes to infinity. Its height is 4, not 5.)

(f) 0.5 pnt The average output power is $R_Y(0) = 4\frac{1}{2}$.

(g) 1.5 pnt Due to independence and the fact that the noise is zero mean, the autocorrelations of $h(t) * X(t)$ and $W(t)$ add up. Furthermore, $R_W(\tau) = 3\delta(\tau)$. Altogether,

$$R_Y(\tau) = \frac{1}{2}e^{-|\tau|} + 4 + 3\delta(\tau)$$

Apply (as before) the Fourier transform:

$$S_Y(f) = \frac{1}{1+(2\pi f)^2} + 4\delta(f) + 3$$

Also here, the PSDs of the signal and the noise add up.