Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Signal Processing Systems

Resit exam EE2S31 SIGNAL PROCESSING
20 July 2023 13:30–16:30

Closed book; two sides of one A4 with handwritten notes permitted. No other tools except a basic pocket calculator permitted. Note the attached tables!

This exam consists of five questions (36 points). Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.
Hint: Avoid losing too much time on detailed calculations, write down the general approach first.

Question 1 (11 points)

We want to estimate an unknown audio channel (i.e. FIR filter) $h[n]$ by sending a known audio signal $x[n]$ through it, and measuring the output $y[n] = x[n] * h[n]$. Our goal is to locate the first peak in $h[n]$.

The length of $x[n]$ is $N_x = 500$. The sample rate of the receiver is 20 kHz. We take $N_y = 2000$ samples of the output signal $y[n]$. Let us assume that the channel impulse response is of length L.

We will estimate $h[n]$ in the frequency domain, using the expression $H(\omega) = \frac{Y(\omega)}{X(\omega)}$. Considering the sampling rate of the receiver, the distance between two consecutive samples of $h[n]$ is 1/20 kHz = 0.05 ms. Therefore, we say that we can locate the first peak with 0.05 ms resolution.

(a) Give the formula for the relationship between N_x, N_y and L and determine the value L.

(b) How can you use the DFT and the inverse DFT (IDFT) to estimate $h[n]$? Explain the steps in detail!

In an alternative time-domain approach, we propose a different estimator

$$\hat{h}_2[n] = \frac{1}{\alpha}\, x[-n] * y[n]$$

where α is a scaling.

(c) Motivate this approach. For this, express $\hat{h}_2[n]$ in terms of $h[n]$ and use terminology such as autocorrelation.

(d) Determine a suitable value for α (motivate).

(e) Compare the computational complexity of both methods. For the frequency-domain approach, assume that we use a radix-2 FFT algorithm to compute the DFT and IDFT. Indicate the number of additions and multiplications needed for each approach.

Returning to the frequency-domain approach, note that there may be frequencies for which $|X(\omega)|$ is very small (let's say smaller than a threshold ϵ). For these frequencies, we simply take $\hat{H}(\omega) = 0$. The channel estimate is then

$$\hat{H}(\omega) = \frac{Y(\omega)}{X(\omega)} G(\omega), \qquad \text{where } G(\omega) = \begin{cases} 1 & \text{if } |X(\omega)| > \epsilon \\ 0 & \text{otherwise} \end{cases}$$

In the noiseless case, this means that $\hat{H}(\omega) = H(\omega)G(\omega)$.

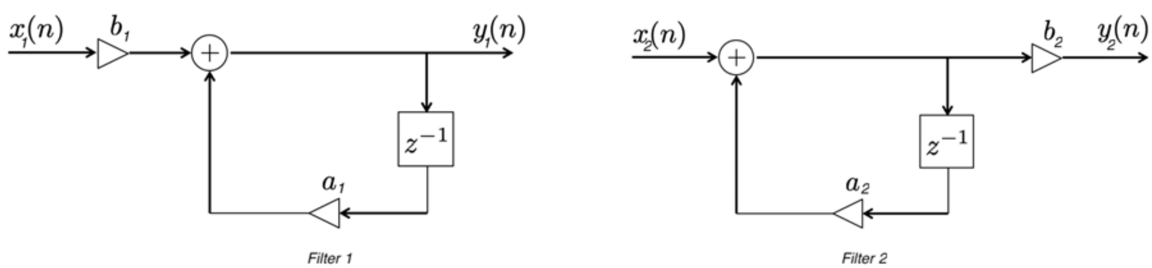
(f) Express $\hat{h}[n]$ in terms of $h[n]$. Given the spectrum of $x[n]$ in the figure below, what is the effect of the thresholding (described above) on the channel estimate $\hat{h}[n]$?

(g) Determine $g[n]$, the inverse DTFT of $G(\omega)$ and make a sketch!

(h) Considering $g[n]$, what is the resolution now for locating the first peak?

Question 2 (8 points)

Consider the following first-order IIR filter realizations.



(a) Assuming that the multipliers $a_1 = a_2$ and the inputs $x_1(n) = x_2(n)$, for which value of the multiplier b_1 does Filter 1 have the same output as Filter 2? Express it in terms of the multiplier b_2!

(b) Give a formula for the transfer function of Filter 1!

(c) Give a formula for the impulse response of the filter in Filter 1!

In a practical scenario we have to quantize the outputs of the multipliers. Assume that the quantizers we use are uniform midtread quantizers with stepsize Δ. Assuming Δ is small enough, the quantization error can be modeled as an additive noise signal $z(n)$, which is a realization of an uncorrelated wide-sense stationary process, having a uniform distribution over the interval $[-\Delta/2, \Delta/2]$.

(d) Compute the total quantization noise power at the output of Filter 1!

(e) Assuming again that $a_1 = a_2$, for which value of b_2 does Filter 2 have the same quantization noise power at the output as Filter 1?

Question 3 (6 points)

A joint probability density function of the random variables X and Y is given by

$$f_{X,Y}(X,Y) = \begin{cases} \frac{x^4}{2} & \text{for } 0 \le x \le 1,\ 0 \le y \le x^2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute the marginal pdfs of the random variables X and Y.

(b) Compute the conditional pdf $f_{X|Y}(x|y)$.

For the remainder of this question, assume that $f_{X|Y}(x|y)$ is given by

$$f_{X|Y}(x|y) = \begin{cases} \frac{x^2}{2(1-y^{3/2})} & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and that $f_X(x)$ is given by

$$f_X(x) = \begin{cases} \frac{5x^4}{2} & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

In an experiment, we observe a realization of random variable Y, while we want to make an estimate of X. To do so, we determine two different estimators for X.

(c) Calculate $\hat{X}_B$, the "blind" estimate.

(d) Calculate $\hat{X}_M(y)$, the MMSE estimate of X given a single observation y of Y.

(e) Explain in words which of the two estimators from Question (c) and (d) is better. Also define what you mean by "better".

(f) Under which conditions are the two estimators in Question (c) and (d) equal?

Question 4 (5 points)

Let Z be a random variable with a Laplace(λ) distribution:

$$f_Z(z) = \frac{\lambda}{2} e^{-\lambda|z|}, \qquad \lambda > 0$$

(a) Draw $f_Z(z)$ for $\lambda = 1$.

(b) By comparing to a Gaussian distribution, explain why a Laplace distribution is often used to model random noise with *outliers* (i.e., occasional large numbers).

(c) Derive that the moment generating function (MGF) of Z is given by

$$\phi_Z(s) = \frac{\lambda^2}{\lambda^2 - s^2}$$

and specify the ROC.

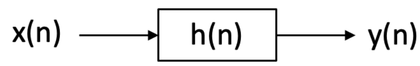
(d) Compute $\mathrm{E}[Z]$ and $\mathrm{var}[Z]$.

Question 5 (6 points)

Consider an application where a signal $x[n]$ is communicated from a transmitting device to a receiving device. Signal $x[n]$ is considered to be a realization of a zero-mean WSS random process $X[n]$ with autocorrelation function

$$R_X[k] = \sigma_X^2 \left(\delta[k+1] + 3\delta[k] + \delta[k-1] \right) .$$

For efficiency, prior to transmission, process $X[n]$ is first decorrelated (or whitened) by a filter with impulse response $h[n]$, leading to a white random process $Y[n]$. Let $H(\phi)$ be the DTFT of $h[n]$.



(a) Determine the power spectral density $S_X(\phi)$.

(b) Give the magnitude-squared response $|H(\phi)|^2$ of a filter that leads to a decorrelated process $Y[n]$ with variance σ_X^2.

Assume now that the impulse response $h[n]$ is given by $h[n] = \delta[n-1]$, while $R_Y[k]$ is now unknown, and $R_X[k]$ is the same as before.

(c) Give the crosscorrelation function $R_{XY}[k]$ and the autocorrelation function $R_Y[k]$ for this situation.

Suppose now that we upsample $x[n]$ by a factor $L = 2$, i.e., $v[n] = [\cdots, 0, \boxed{x[0]}, 0, x[1], 0, x[2], 0, \cdots]$. Let $V[n]$ be the corresponding random signal.

(d) Determine the autocorrelation function $R_V[n, k]$. Is $V[n]$ WSS?

Discrete Time function	Discrete Time Fourier Transform
$\delta[n] = \delta_n$	1
1	$\delta(\phi)$
$\delta[n-n_0] = \delta_{n-n_0}$	$e^{-j2\pi\phi n_0}$
$u[n]$	$\dfrac{1}{1-e^{-j2\pi\phi}} + \dfrac{1}{2}\sum_{k=-\infty}^{\infty}\delta(\phi+k)$
$e^{j2\pi\phi_0 n}$	$\sum_{k=-\infty}^{\infty}\delta(\phi-\phi_0-k)$
$\cos 2\pi\phi_0 n$	$\frac{1}{2}\delta(\phi-\phi_0)+\frac{1}{2}\delta(\phi+\phi_0)$
$\sin 2\pi\phi_0 n$	$\frac{1}{2j}\delta(\phi-\phi_0)-\frac{1}{2j}\delta(\phi+\phi_0)$
$a^n u[n]$	$\dfrac{1}{1-ae^{-j2\pi\phi}}$
$a^{\lvert n\rvert}$	$\dfrac{1-a^2}{1+a^2-2a\cos 2\pi\phi}$
g_{n-n_0}	$G(\phi)e^{-j2\pi\phi n_0}$
$g_n e^{j2\pi\phi_0 n}$	$G(\phi-\phi_0)$
g_{-n}	$G^*(\phi)$
$\sum_{k=-\infty}^{\infty} h_k g_{n-k}$	$G(\phi)H(\phi)$
$g_n h_n$	$\int_{-1/2}^{1/2} H(\phi')G(\phi-\phi')\,d\phi'$

Note that $\delta[n]$ is the discrete impulse, $u[n]$ is the discrete unit step, and a is a constant with magnitude $|a| < 1$.

Table 3 Discrete-Time Fourier transform pairs and properties.

TABLE 4.5 Properties of the Fourier Transform for Discrete-Time Signals

Property	Time Domain	Frequency Domain
Notation	$x(n)$	$X(\omega)$
	$x_1(n)$	$X_1(\omega)$
	$x_2(n)$	$X_2(\omega)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time shifting	$x(n-k)$	$e^{-j\omega k}X(\omega)$
Time reversal	$x(-n)$	$X(-\omega)$
Convolution	$x_1(n) * x_2(n)$	$X_1(\omega)X_2(\omega)$
Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega)$
		$= X_1(\omega)X_2^*(\omega)$
		[if $x_2(n)$ is real]
Wiener–Khintchine theorem	$r_{xx}(l)$	$S_{xx}(\omega)$
Frequency shifting	$e^{j\omega_0 n}x(n)$	$X(\omega - \omega_0)$
Modulation	$x(n)\cos\omega_0 n$	$\frac{1}{2}X(\omega + \omega_0) + \frac{1}{2}X(\omega - \omega_0)$
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega - \lambda)d\lambda$
Differentiation in		
the frequency domain	$nx(n)$	$j\frac{dX(\omega)}{d\omega}$
Conjugation	$x^*(n)$	$X^*(-\omega)$
Parseval's theorem	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega$	