Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Circuits and Systems

Partial exam EE2S31 SIGNAL PROCESSING
Part 1: 17 May 2022 (13:30-15:30)

Closed book; two sides of one A4 with handwritten notes permitted. No other tools except a basic pocket calculator permitted.

This exam consists of four questions (34 points). Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.
Hint: Avoid losing too much time on detailed calculations, write down the general approach first.

Question 1 (11 points)

Suppose I catch a cold. X is the time until I infect someone else. As part of that interval, let Y be the incubation period. We model this as follows:

Random variable X has a second-order Erlang PDF:

$$f_X(x) = \begin{cases} \lambda^2 \, x \, e^{-\lambda x} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(with $\lambda > 0$). Given $X = x$, Y is a Uniform$(0, x)$ random variable.

(a) What is $f_{Y|X}(y|x)$.

(b) What is $f_{X,Y}(x, y)$.

(c) What is $f_Y(y)$.

(d) Derive that

$$f_{X|Y}(x|y) = \begin{cases} \lambda e^{-\lambda(x-y)} & x \geq y, \\ 0 & \text{otherwise.} \end{cases}$$

(e) Compute $\mathrm{E}[X]$, $\mathrm{E}[Y]$ and $\mathrm{cov}[X, Y]$.

(f) Find $\hat{x}_{\mathrm{MMSE}}(y)$, the MMSE estimate of X given $Y = y$.

(g) Find $\hat{x}_{\mathrm{ML}}(y)$, the Maximum Likelihood estimate of X given $Y = y$.

(h) Use the Chebyshev inequality to find an upper bound for $\mathrm{P}[X \geq 4/\lambda]$.

(i) Find the PDF of $W = X - Y$.

From Appendix A

Exponential (λ)

For $\lambda > 0$,

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad \phi_X(s) = \frac{\lambda}{\lambda - s}$$

$$\mathrm{E}[X] = 1/\lambda$$
$$\mathrm{Var}[X] = 1/\lambda^2$$

Erlang (n, λ)

For $\lambda > 0$, and a positive integer n,

$$f_X(x) = \begin{cases} \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad \phi_X(s) = \left(\frac{\lambda}{\lambda - s}\right)^n$$

$$\mathrm{E}[X] = n/\lambda$$
$$\mathrm{Var}[X] = n/\lambda^2$$

Solution

(a) 1 pnt $f_{Y|X}(y|x)$ is the uniform distribution:

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{x} & 0 \leq y \leq x \\ 0 & \text{otherwise.} \end{cases}$$

(b) 1 pnt

$$f_{X,Y}(x,y) = f_{Y|X}(y|x) f_X(x) = \begin{cases} \lambda^2 e^{-\lambda x} & 0 \leq y \leq x,\ x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(c) 1 pnt

$$f_Y(y) = \int f_{X,Y}(x,y)\mathrm{d}x = \int_y^\infty \lambda^2 e^{-\lambda x}\mathrm{d}x = \begin{cases} \lambda e^{-\lambda y} & y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

which is an exponential distribution.

(d) 1 pnt

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \lambda e^{-\lambda(x-y)} & x \geq y \\ 0 & \text{otherwise.} \end{cases}$$

which is a shifted exponential distribution.

(e) 2 pnt From the mean of the Erlang distribution resp. the exponential distribution,

$$\mathrm{E}[X] = \frac{2}{\lambda}, \qquad \mathrm{E}[Y] = \frac{1}{\lambda}$$

$$\begin{aligned} \mathrm{E}[XY] &= \int_0^\infty \int_0^x xy\lambda^2 e^{-\lambda x}\mathrm{d}y\mathrm{d}x \\ &= \int_0^\infty \lambda^2 x e^{-\lambda x} \left[\frac{1}{2}y^2\right]_0^x \mathrm{d}x \\ &= \int_0^\infty \frac{1}{2}\lambda^2 x^3 e^{-\lambda x}\mathrm{d}x \\ &= \frac{3}{\lambda^2} \end{aligned}$$

The final step follows from the expression of the mean of an Erlang distribution with $n = 3$.

$$\mathrm{cov}[X,Y] = \mathrm{E}[XY] - \mathrm{E}[X]\mathrm{E}[Y] = \frac{3}{\lambda^2} - \frac{2}{\lambda}\frac{1}{\lambda} = \frac{1}{\lambda^2}$$

(f) 1 pnt Using properties of the exponential distribution,

$$\begin{aligned}\hat{x}_{\text{MMSE}}(y) = \text{E}[X|Y=y] &= \int_y^\infty \lambda x e^{-\lambda(y-x)} \text{d}x \\ &= \int_0^\infty \lambda(y+u)e^{-\lambda u}\text{d}u \\ &= y\int_0^\infty \lambda e^{-\lambda u}\text{d}u + \int_0^\infty u\,\lambda e^{-\lambda u}\text{d}u \\ &= y + \frac{1}{\lambda}\end{aligned}$$

(g) 1 pnt The ML is found by optimizing the likelihood function, which is $f_{Y|X}(y|x)$ viewed as function of x (and for a given observation y):

$$\hat{x}_{\text{ML}}(y) = \arg\max_x f_{Y|X}(y|x) = \arg\max_{x,\, x\geq y} \frac{1}{x} = y$$

since the maximum is achieved for $x = y$.

(h) 1 pnt

$$\text{P}\left[X \geq \frac{4}{\lambda}\right] = \text{P}\left[X - \frac{2}{\lambda} \geq \frac{2}{\lambda}\right] = \text{P}\left[|X - \frac{2}{\lambda}| \geq \frac{2}{\lambda}\right] \leq \frac{\text{var}[X]}{c^2}$$

with $c = \frac{2}{\lambda}$ and, for an Erlang distribution with $n = 2$, $\text{var}[X] = \frac{2}{\lambda^2}$, so that

$$\text{P}\left[X \geq \frac{4}{\lambda}\right] \leq \frac{1}{2}$$

(i) 2 pnt Note $y = x - w$, with $y \geq 0 \Rightarrow x \geq w$ and $y \leq x \Rightarrow w \geq 0$. Hence

$$f_W(w) = \int f_{X,Y}(x, x-w)\text{d}x = \int_w^\infty \lambda^2 e^{-\lambda x}\text{d}x = \lambda e^{-\lambda w}$$

for $w \geq 0$, and 0 otherwise. This is an exponential distribution. (Use of the MGF is tricky here, since it is used for the sum of *independent* random variables.)

Alternative derivation:

$$f_W(w) = \int f_{X,Y}(w+y, y)\text{d}y = \int_0^\infty \lambda^2 e^{-\lambda(w+y)}\text{d}y$$

which evaluates to the same result.

W denotes the time from when I am infectious, until I infect someone. We know that Erlang(2) is the sum of two independent exponential RVs (here: $X = Y + W$), so this checks out.

Question 2 (7 points)

A Laplace distribution with scale parameter $a > 0$ has moment generating function (MGF)

$$\phi_Z(s) = \frac{a^2}{a^2 - s^2}, \qquad \text{ROC: } |s| < a$$

(a) Compute $\text{E}[Z]$ and $\text{var}[Z]$ using the MGF.

For $\lambda > 0$, let

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

be the PDF of an exponentially distributed random variable X.

(b) Derive that the moment generating function (MGF) $\phi_X(s)$ is

$$\phi_X(s) = \frac{\lambda}{\lambda - s}, \qquad \text{ROC: } s < \lambda$$

(c) Let $Y = -X$. Derive the MGF of Y.

(d) Derive how we can generate a random variable Z with a Laplace(a) distribution using two independent exponentially distributed random variables X and Y; also specify the parameter λ.

Solution

(a) 2 pnt

$$\begin{aligned}
\frac{\mathrm{d}\phi_Z(s)}{\mathrm{d}s} &= \frac{2a^2 s}{(a^2 - s^2)^2} \\
\frac{\mathrm{d}^2\phi_Z(s)}{\mathrm{d}s^2} &= \frac{-4a^2 s}{(a^2 - s^2)^3} + \frac{2a^2}{(a^2 - s^2)^2} \\
\mathrm{E}[Z] &= \left.\frac{\mathrm{d}\phi_Z(s)}{\mathrm{d}s}\right|_{s=0} = 0 \\
\mathrm{E}[Z^2] &= \left.\frac{\mathrm{d}^2\phi_Z(s)}{\mathrm{d}s^2}\right|_{s=0} = \frac{2a^2}{a^4} = \frac{2}{a^2} \\
\mathrm{var}[Z] &= \mathrm{E}[Z^2] - (\mathrm{E}[Z])^2 = \frac{2}{a^2}
\end{aligned}$$

(b) 2 pnt From the definition,

$$\begin{aligned}
\phi_X(s) = \mathrm{E}[e^{sX}] &= \int_0^\infty e^{sx} \lambda e^{-\lambda x} \mathrm{d}x \\
&= \int_0^\infty \lambda e^{(s-\lambda)x} \mathrm{d}x \\
&= \frac{\lambda}{s - \lambda} \left[e^{(s-\lambda)x} \right]_0^\infty \\
&= \frac{\lambda}{\lambda - s}
\end{aligned}$$

with (from the 2nd equation) ROC: $s < \lambda$.

(c) 1 pnt From the scaling properties (Thm. 9.5),

$$\phi_Y(s) = \phi_X(-s) = \frac{\lambda}{\lambda + s}$$

with ROC: $-s < \lambda \Leftrightarrow s > -\lambda$.

Alternative derivation:

$$f_Y(y) = \begin{cases} \lambda e^{\lambda y} & y \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
\phi_Y(s) &= \int_{-\infty}^{0} \lambda e^{\lambda y} e^{sy} \mathrm{d}y \\
&= \left[\frac{\lambda}{\lambda + s} e^{(\lambda+s)y} \right]_{-\infty}^{0} \\
&= \frac{\lambda}{\lambda + s}
\end{aligned}$$

with (from the 2nd equation) ROC: $s > -\lambda$.

(d) 2 pnt This is found by inspection. Consider $Z = X - Y$, for independent X and Y, both exponentially distributed, then the MGF of Z is

$$\phi_Z(s) = \phi_X(s) \cdot \phi_Y(-s) = \frac{\lambda}{\lambda - s} \frac{\lambda}{\lambda + s} = \frac{\lambda^2}{\lambda - s^2}$$

Thus, Z is Laplace distributed, with $a = \lambda$.

The ROC of the MGF of X is $s < \lambda$, the ROC of the MGF of $-Y$ is $s > -\lambda$, so the ROC of the product is (intersection) $|s| < \lambda$, which matches with the ROC of the MGF of the Laplace.

We have shown that a variable with a Laplace distribution can be generated from the difference of two exponentially distributed RVs.

Question 3 (8 points)

A Barlett window of length $2N-1$ is defined as

$$b_{2N-1}[n] = \begin{cases} \frac{N-|n|}{N} & 0 \le |n| \le N \\ 0 & \text{otherwise} \end{cases}$$

The Barlett window (up to a scaling) can be created by convolving the rectangular window

$$w_N[n] = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

with its time-reversed version $w_N[-n]$.

(1 p) (a) Give the exact expression (including scaling) for the $2N-1$ long Bartlett window in terms of $w_N[n]$.

(2 p) (b) Verify that the Bartlett window can be expressed in the frequency domain as

$$B_{2N-1}(\omega) = \frac{1}{N}\left(\frac{\sin(\omega N/2)}{\sin(\omega/2)}\right)^2 .$$

Hint: recall that the DTFT of $w_N[n]$ is $W_N(\omega) = \frac{\sin(\omega N/2)}{\sin(\omega/2)} \cdot e^{-j\omega(N-1)/2}$.

(2 p) (c) Compare the DTFT magnitude of the length N rectangular window and $2N-1$ Bartlett window using a sketch! Which of the above windows result in less spectral leakage and why?

(1 p) (d) In case we can collect exactly M samples of a given signal, which window can achieve better spectral resolution, a length M Bartlett or length M rectangular window?

(2 p) (e) What is the 4-point DFT of the sequence

$$x[n] = \begin{cases} 1, & n \ge 0 \\ 0, & n < 0 \end{cases}$$

observed through the Bartlett b_{2N-1} for $N = 4$?

Solution

See figure 1 for the time-domain representation of the Bartlett window.

(a)

$$b_{2N-1}[m] = \frac{1}{N}\sum_{n=0}^{N-1} w[n]w[m-(-n)], \quad m = -(N-1) \cdots, N-1$$

The scalar $\frac{1}{N}$ is needed to obtain $b_{2N-1}[0] = 1$ instead of $b_{2N-1}[0] = N$ which is the result of the convolution without scaling.

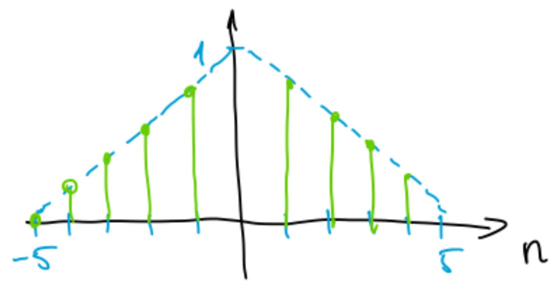

Figure 1

(b) Convolution in the time domain is multiplication in the DTFT domain, therefore (using the time-reversal property of the Fourier Transform):

$$
\begin{aligned}
B_{2N-1}[\omega] = \mathcal{F}\{b_{2N-1}[n]\} &= \frac{1}{N}\mathcal{F}\{w[n]\} \cdot \mathcal{F}\{w[-n]\} = \\
&= \frac{1}{N} W_N(\omega) \cdot W_N(-\omega) = \frac{1}{N}\frac{\sin(\omega N/2)}{\sin(\omega/2)} \cdot e^{-j\omega(N-1)/2} \cdot \frac{\sin(-\omega N/2)}{\sin(-\omega/2)} \cdot e^{j\omega(N-1)/2} = \\
&= \frac{1}{N}\left(\frac{\sin(\omega N/2)}{\sin(\omega/2)}\right)^2
\end{aligned}
$$

(c) See Figure 2.

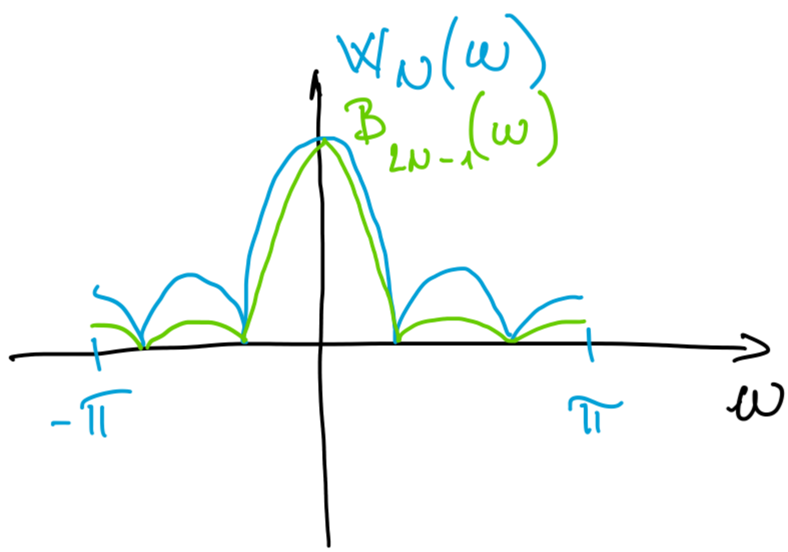

Figure 2

The Barlett window has less specral leakage due to the smaller sidelobes.

(d) Spectral resolution depends on the width of the main lobe. The main lobe of the $2N-1$ Barlett window is the same as that of the $N-1$ rectangular window. However, for the same length, $2N-1$, the rectangular window would have a narrower main lobe. So, the rectangular window will achieve a better resolution.

(e) The observed sequence $y[n] = [1, 0.75, 0.5, 0.25, 0]$. Its 4-point DFT is computed as

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0.75 \\ 0.5 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 0.5 - 0.5j \\ 0.5 \\ 0.5 + 0.5j \end{bmatrix}$$

Question 4 (8 points)

Our antenna receives two radio signals at the same time, $X_a^{(1)}(F)$ and $X_a^{(2)}(F)$, where $X_a^{(1)}(F) \neq 0$ for $40 < |F| < 50$ and $X_a^{(2)}(F) \neq 0$ for $50 < |F| < 60$.

Our radio receiver first samples the signal $X_a(F) = X_a^{(1)}(F) + X_a^{(2)}(F)$ and then applies filters to separate the two radio signals from each other.

(2 p) (a) Sketch the spectrum (in the interval -60 to 60 Hz) of the digital signal $X(F)$ in case we choose a sampling rate $F_s = 40$ Hz.

(1 p) (b) Define the ideal digital low-pass filter that extracts the baseband copy of $X^{(1)}$ from $X^{(F)}$ (i.e., the spectral copy of $X^{(1)}(F)$ which is closest to 0 Hz). Let us call this baseband copy $X_B^{(1)}(F)$.

(2 p) (c) We will use zero-order hold interpolation to construct the analog version of $X_B^{(1)}$. The impulse response of the zero-order interpolation filter is given by

$$h_0(t) = u(t) - u(t - T) = \begin{cases} 1, & 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$

Prove that the frequency response of this interpolation filter in the frequency domain is

$$H_0(\Omega) = e^{-j\Omega T/2}\, T\, \frac{\sin(\Omega T/2)}{\Omega T/2}\,.$$

(2 p) (d) Sketch the magnitude impulse reponse $|H_0(\Omega)|$ along with the magnitude impulse response of the ideal interpolation filter and compare. Can $H_0(\Omega)$ give you a perfect reconstruction of $x_B^{(1)}$? If not, at which frequencies will you observe distortion?

(1 p) (e) Define a digital filter, to be applied prior to interpolation, which compensates for the zero-order-hold interpolation!

Solution

(a) See Figure 3.

(b) $L(F) = 1$ for $0 \leq |F| \leq 10$ and 0 otherwise. See also Figure 4.

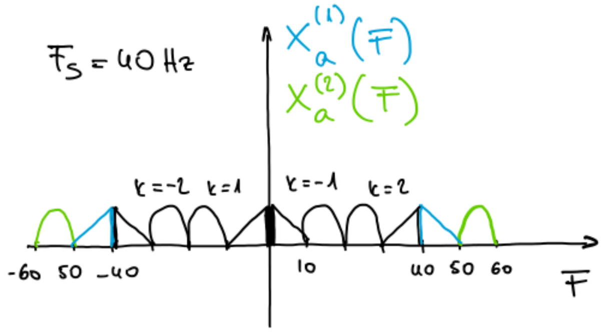

Figure 3

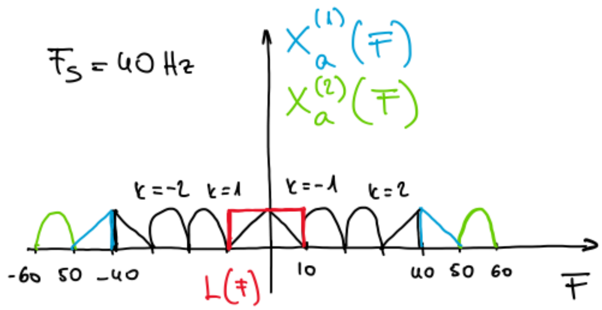

Figure 4

(c)

$$H_0(\Omega) = \int_{-\infty}^{\infty} (u(t) - u(t-T))\, e^{-j\Omega t} dt =$$
$$= \int_0^T e^{-j\Omega t} dt = -\frac{1}{j\Omega} e^{-j\Omega t}\Big|_0^T = \frac{e^{-j\Omega T} - e^0}{-j\Omega} = \frac{e^{-j\Omega T/2}(e^{j\Omega T/2} - e^{-j\Omega T/2})}{j\Omega} =$$
$$= e^{-j\Omega T/2} \frac{2j \sin(\Omega T/2)}{j\Omega} = e^{-j\Omega T/2} T \frac{\sin(\Omega T/2)}{\Omega T/2}$$

(d) See Figure 5. The ideal interpolation filter $|H_i(\Omega)|$ is in blue, while $|H_0(\Omega)|$ is in green. The reconstruction of $X_B^{(1)}$ will have attenuated frequencies up to 10 Hz. Above 10 Hz the signal has 0 frequency content due to filtering in part b).

(e) See Figure 5. The compenation filter $H_c(\Omega)$ is in red.

$$H_c(\Omega) = \begin{cases} \frac{1}{H_0(\Omega)} & \text{for } \Omega > \Omega_s/2 = 20\text{Hz} \\ 0 & \text{otherwise} \end{cases}$$

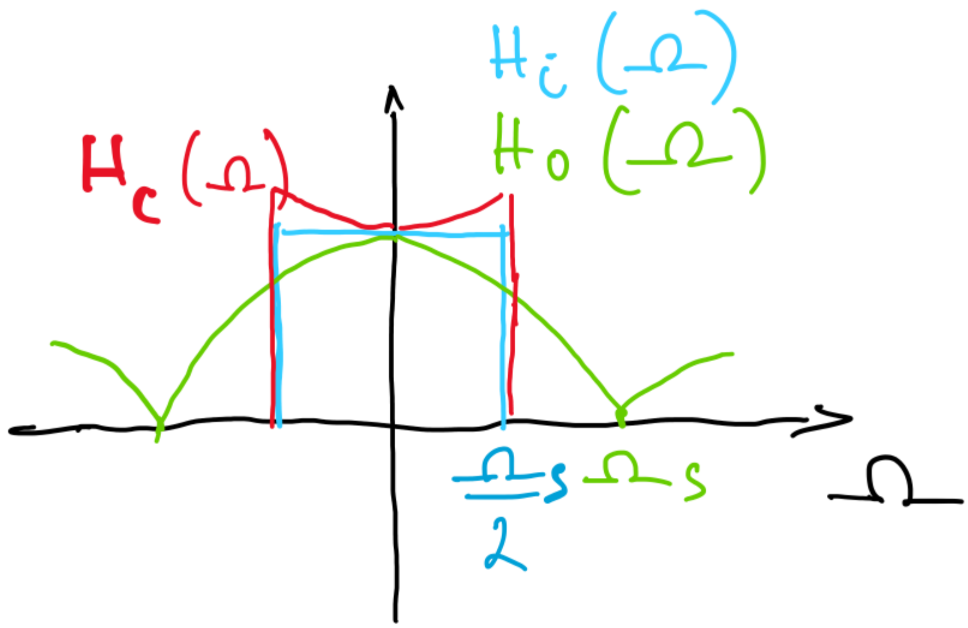

Figure 5