Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Circuits and Systems

Partial exam EE2S31 SIGNAL PROCESSING
Part 1: 17 May 2022 (13:30-15:30)

Closed book; two sides of one A4 with handwritten notes permitted. No other tools except a basic pocket calculator permitted.

This exam consists of four questions (35 points). Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.
Hint: Avoid losing too much time on detailed calculations, write down the general approach first.

Question 1 (11 points)

Suppose I catch a cold. X is the time until I infect someone else. As part of that interval, let Y be the incubation period. We model this as follows:

Random variable X has a second-order Erlang PDF:

$$f_X(x) = \begin{cases} \lambda^2 \, x \, e^{-\lambda x} & x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(with $\lambda > 0$). Given $X = x$, Y is a Uniform$(0, x)$ random variable.

(a) What is $f_{Y|X}(y|x)$.

(b) What is $f_{X,Y}(x, y)$.

(c) What is $f_Y(y)$.

(d) Derive that

$$f_{X|Y}(x|y) = \begin{cases} \lambda e^{-\lambda(x-y)} & x \geq y, \\ 0 & \text{otherwise.} \end{cases}$$

(e) Compute E$[X]$, E$[Y]$ and cov$[X, Y]$.

(f) Find $\hat{x}_{\text{MMSE}}(y)$, the MMSE estimate of X given $Y = y$.

(g) Find $\hat{x}_{\text{ML}}(y)$, the Maximum Likelihood estimate of X given $Y = y$.

(h) Use the Chebyshev inequality to find an upper bound for P$[X \geq 4/\lambda]$.

(i) Find the PDF of $W = X - Y$.

From Appendix A

Exponential (λ)

For $\lambda > 0$,

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad \phi_X(s) = \frac{\lambda}{\lambda - s}$$

$$\text{E}[X] = 1/\lambda$$
$$\text{Var}[X] = 1/\lambda^2$$

Erlang (n, λ)

For $\lambda > 0$, and a positive integer n,

$$f_X(x) = \begin{cases} \dfrac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad \phi_X(s) = \left(\frac{\lambda}{\lambda - s}\right)^n$$

$$\mathrm{E}[X] = n/\lambda$$

$$\mathrm{Var}[X] = n/\lambda^2$$

Question 2 (7 points)

A Laplace distribution with scale parameter $a > 0$ has moment generating function (MGF)

$$\phi_Z(s) = \frac{a^2}{a^2 - s^2}, \qquad \text{ROC: } |s| < a$$

(a) Compute $\mathrm{E}[Z]$ and $\mathrm{var}[Z]$ using the MGF.

For $\lambda > 0$, let

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

be the PDF of an exponentially distributed random variable X.

(b) Derive that the moment generating function (MGF) $\phi_X(s)$ is

$$\phi_X(s) = \frac{\lambda}{\lambda - s}, \qquad \text{ROC: } s < \lambda$$

(c) Let $Y = -X$. Derive the MGF of Y.

(d) Derive how we can generate a random variable Z with a Laplace(a) distribution using two independent exponentially distributed random variables X and Y; also specify the parameter λ.

Question 3 (9 points)

A Barlett window of length $2N-1$ is defined as

$$b_{2N-1}[n] = \begin{cases} \frac{N-|n|}{N} & 0 \le |n| \le N \\ 0 & \text{otherwise} \end{cases}$$

The Barlett window (up to a scaling) can be created by convolving the rectangular window

$$w_N[n] = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

with its time-reversed version $w_N[-n]$.

(1 p) (a) Give the exact expression (including scaling) for the $2N-1$ long Bartlett window in terms of $w_N[n]$.

(2 p) (b) Verify that the Bartlett window can be expressed in the frequency domain as

$$B_{2N-1}(\omega) = \frac{1}{N}\left(\frac{\sin(\omega N/2)}{\sin(\omega/2)}\right)^2 .$$

Hint: recall that the DTFT of $w_N[n]$ is $W_N(\omega) = \frac{\sin(\omega N/2)}{\sin(\omega/2)} \cdot e^{-j\omega(N-1)/2}$.

(2 p) (c) Compare the DTFT magnitude of the length N rectangular window and $2N-1$ Bartlett window using a sketch! Which of the above windows result in less spectral leakage and why?

(1 p) (d) In case we can collect exactly M samples of a given signal, which window can achieve better spectral resolution, a length M Bartlett or length M rectangular window?

(2 p) (e) What is the 4-point DFT of the sequence

$$x[n] = \begin{cases} 1, & n \ge 0 \\ 0, & n < 0 \end{cases}$$

observed through the Bartlett b_{2N-1} for $N=4$?

Question 4 (8 points)

Our antenna receives two radio signals at the same time, $X_a^{(1)}(F)$ and $X_a^{(2)}(F)$, where $X_a^{(1)}(F) \neq 0$ for $40 < |F| < 50$ and $X_a^{(2)}(F) \neq 0$ for $50 < |F| < 60$.

Our radio receiver first samples the signal $X_a(F) = X_a^{(1)}(F) + X_a^{(2)}(F)$ and then applies filters to separate the two radio signals from each other.

(2 p) (a) Sketch the spectrum (in the interval -60 to 60 Hz) of the digital signal $X(F)$ in case we choose a sampling rate $F_s = 40$ Hz.

(1 p) (b) Define the ideal digital low-pass filter that extracts the baseband copy of $X^{(1)}$ from $X^{(F)}$ (i.e., the spectral copy of $X^{(1)}(F)$ which is closest to 0 Hz). Let us call this baseband copy $X_B^{(1)}(F)$.

(2 p) (c) We will use zero-order hold interpolation to construct the analog version of $X_B^{(1)}$. The impulse response of the zero-order interpolation filter is given by

$$h_0(t) = u(t) - u(t-T) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

Prove that the frequency response of this interpolation filter in the frequency domain is

$$H_0(\Omega) = e^{-j\Omega T/2}\, T \frac{\sin(\Omega T/2)}{\Omega T/2}.$$

(2 p) (d) Sketch the magnitude impulse reponse $|H_0(\Omega)|$ along with the magnitude impulse response of the ideal interpolation filter and compare. Can $H_0(\Omega)$ give you a perfect reconstruction of $x_B^{(1)}$? If not, at which frequencies will you observe distortion?

(1 p) (e) Define a digital filter, to be applied prior to interpolation, which compensates for the zero-order-hold interpolation!

TABLE 4.5 Properties of the Fourier Transform for Discrete-Time Signals

Property	Time Domain	Frequency Domain
Notation	$x(n)$	$X(\omega)$
	$x_1(n)$	$X_1(\omega)$
	$x_2(n)$	$X_2(\omega)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time shifting	$x(n-k)$	$e^{-j\omega k}X(\omega)$
Time reversal	$x(-n)$	$X(-\omega)$
Convolution	$x_1(n) * x_2(n)$	$X_1(\omega)X_2(\omega)$
Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega)$ $= X_1(\omega)X_2^*(\omega)$ [if $x_2(n)$ is real]
Wiener–Khintchine theorem	$r_{xx}(l)$	$S_{xx}(\omega)$
Frequency shifting	$e^{j\omega_0 n}x(n)$	$X(\omega - \omega_0)$
Modulation	$x(n)\cos\omega_0 n$	$\frac{1}{2}X(\omega+\omega_0) + \frac{1}{2}X(\omega-\omega_0)$
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega-\lambda)d\lambda$
Differentiation in the frequency domain	$nx(n)$	$j\frac{dX(\omega)}{d\omega}$
Conjugation	$x^*(n)$	$X^*(-\omega)$
Parseval's theorem	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega$	