Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Circuits and Systems

Resit exam EE2S31 SIGNAL PROCESSING
20 July 2022 (13:30-16:30)

Closed book; two sides of one A4 with handwritten notes permitted. No other tools except a basic pocket calculator permitted.

This exam consists of five questions (36 points). Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.
Hint: Avoid losing too much time on detailed calculations, write down the general approach first.

Question 1 (7 points)

Let Y be a random variable with PDF

$$f_Y(y) = \begin{cases} c(y+1) & -1 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

where c is a constant. Given $Y = y$, the random variable X is uniformly distributed between -1 and y.

(a) Find the constant c.

(b) Find the conditional PDF, $f_{X|Y}(x|y)$.

(c) Find the joint PDF, $f_{X,Y}(x,y)$, and make a diagram that shows the support of $f_{X,Y}(x,y)$.

(d) Find $\hat{X}_M(Y)$, the minimum mean square error (MMSE) estimator for X given a single sample of Y.

(e) Find $\hat{Y}_{\text{ML}}(X)$, the maximum likelihood estimator for Y given a single sample of X.

Question 2 (6 points)

For this question you might want to make use of Table 4.2, included at the end of this exam.

Random variables $X_1, X_2, \cdots$ are iid, with mean value $\mu = 75$ and standard deviation $\sigma = 15$. Let

$$M_n(X) = \frac{X_1 + \cdots + X_n}{n}$$

be the sample mean.

(a) How many samples n do we need to guarantee that $M_n(X)$ is between 74 and 76 with probability larger than 0.99?

[Hint: use Chebyshev]

(b) If we also know that each X_i is Gaussian distributed, then how many samples n would we need to expect that $M_n(X)$ is between 74 and 76 with probability larger than 0.99?

[Hint: use the CLT]

Suppose now that the X_i are not iid but correlated, such that

$$\begin{cases} \text{var}[X_i] = \sigma^2 & i = 1, \cdots, n \\ \text{cov}[X_i, X_{i+1}] = \sigma^2 a & i = 1, \cdots, n-1 \\ \text{cov}[X_i, X_j] = 0 & \text{otherwise} \end{cases} \tag{1}$$

where a is a constant such that $|a| < 1$.

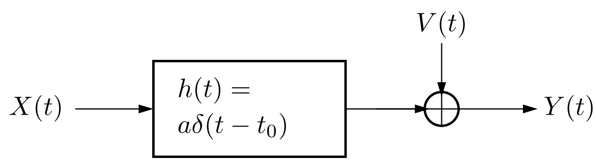
(c) Show that $\text{var}[X_1 + \cdots + X_n] = \sigma^2[n + 2(n-1)a]$.

(d) Let $a = 0.8$. Repeat part (a): How many samples n do we now need to guarantee that $M_n(X)$ is between 74 and 76 with probability larger than 0.99?

Question 3 (6 points)

For this question you might want to make use of Table 1,2, included at the end of this exam.

In EPO4, we use an audio beacon to estimate a propagation delay. In this question we consider a simple model for this.



Let the beacon signal $X(t)$ be a zero mean wide-sense stationary (WSS) signal, with auto-correlation function

$$R_X(\tau) = \frac{\sin(2000\pi\tau)}{2000\pi\tau}.$$

Let $V(t)$ be white Gaussian noise with variance σ_V^2. The audio channel is simply modeled with an attenuation a and a delay t_0. The received signal is

$$Y(t) = aX(t - t_0) + V(t)$$

Thus, the channel is represented by a filter with impulse reponse $h(t) = a\delta(t - t_0)$.

(a) Determine the auto-correlation function $R_Y(t, \tau)$.

(b) Determine the cross-correlation function $R_{XY}(t, \tau)$.

(c) Is $Y(t)$ WSS?

(d) Are $X(t), Y(t)$ jointly WSS?

(e) Determine the input power spectral density $S_X(f)$.

(f) Determine the output power spectral density $S_Y(f)$ and the cross power spectral density $S_{XY}(f)$.

(g) Given (estimates of) $R_X(\tau)$, $R_{XY}(t, \tau)$ and $R_Y(t, \tau)$, how can we estimate t_0?

Question 4 (10 points)

2p (a) Write down the 4-point DFT matrix, and explain how the elements of the matrix are related to the fourth roots of unity using a sketch on the unit circle.

2p (b) Show (using any method you prefer) that the DFT of $x[n] = \begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix}$ is $X[k] = \begin{bmatrix}10 & -2+2j & -2 & -2-2j\end{bmatrix}$.

2p (c) Compute the 8-point DFT of the sequence $y[n] = \begin{bmatrix}1 & 0 & 2 & 0 & 3 & 0 & 4 & 0\end{bmatrix}$ using a linear combination of two 4-point DFTs.

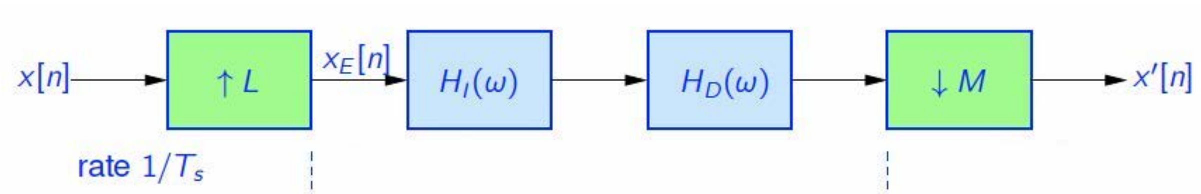
2p (d) The 8-point decimation-in-time algorithm is illustrated in a butterfly diagram in Figure 1 on the Answer Sheet, included at the end of this exam.

Use now this diagram to compute the 8-point DFT of $y[n]$. Specifically, indicate all intermediate results in the signal flow graph and hand it in as your solution.

2p (e) How many butterflies would the radix-2 FFT algorithm have for the computation of a 32-point DFT?

Question 5 (7 points)

Given the following conversion rate system with $L = 3$, $M = 2$, and $T_s = 10$ms.



1p (a) What are the data rates of the signals $x[n]$, $x_E[n]$ and $x'[n]$?

2p (b) What are the roles of the filters $H_I(\omega)$ and $H_D(\omega)$?

2p (c) Can you unite these filters into one filter? If yes, give the specification of the resulting single (ideal) filter in terms of its cut-off frequency. If not, explain why!

2p (d) Develop an alternative implementation of this system, where the filter(s) run at 100Hz data rate, using a polyphase representation and a noble identity. Make two sketches! In the first sketch, draw the modified system after using the polyphase representation. In the second sketch, draw the system after applying the appropriate noble identity as well!

* * *

z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$
0.00	0.5000	0.50	0.6915	1.00	0.8413	1.50	0.9332	2.00	0.97725	2.50	0.99379
0.01	0.5040	0.51	0.6950	1.01	0.8438	1.51	0.9345	2.01	0.97778	2.51	0.99396
0.02	0.5080	0.52	0.6985	1.02	0.8461	1.52	0.9357	2.02	0.97831	2.52	0.99413
0.03	0.5120	0.53	0.7019	1.03	0.8485	1.53	0.9370	2.03	0.97882	2.53	0.99430
0.04	0.5160	0.54	0.7054	1.04	0.8508	1.54	0.9382	2.04	0.97932	2.54	0.99446
0.05	0.5199	0.55	0.7088	1.05	0.8531	1.55	0.9394	2.05	0.97982	2.55	0.99461
0.06	0.5239	0.56	0.7123	1.06	0.8554	1.56	0.9406	2.06	0.98030	2.56	0.99477
0.07	0.5279	0.57	0.7157	1.07	0.8577	1.57	0.9418	2.07	0.98077	2.57	0.99492
0.08	0.5319	0.58	0.7190	1.08	0.8599	1.58	0.9429	2.08	0.98124	2.58	0.99506
0.09	0.5359	0.59	0.7224	1.09	0.8621	1.59	0.9441	2.09	0.98169	2.59	0.99520
0.10	0.5398	0.60	0.7257	1.10	0.8643	1.60	0.9452	2.10	0.98214	2.60	0.99534
0.11	0.5438	0.61	0.7291	1.11	0.8665	1.61	0.9463	2.11	0.98257	2.61	0.99547
0.12	0.5478	0.62	0.7324	1.12	0.8686	1.62	0.9474	2.12	0.98300	2.62	0.99560
0.13	0.5517	0.63	0.7357	1.13	0.8708	1.63	0.9484	2.13	0.98341	2.63	0.99573
0.14	0.5557	0.64	0.7389	1.14	0.8729	1.64	0.9495	2.14	0.98382	2.64	0.99585
0.15	0.5596	0.65	0.7422	1.15	0.8749	1.65	0.9505	2.15	0.98422	2.65	0.99598
0.16	0.5636	0.66	0.7454	1.16	0.8770	1.66	0.9515	2.16	0.98461	2.66	0.99609
0.17	0.5675	0.67	0.7486	1.17	0.8790	1.67	0.9525	2.17	0.98500	2.67	0.99621
0.18	0.5714	0.68	0.7517	1.18	0.8810	1.68	0.9535	2.18	0.98537	2.68	0.99632
0.19	0.5753	0.69	0.7549	1.19	0.8830	1.69	0.9545	2.19	0.98574	2.69	0.99643
0.20	0.5793	0.70	0.7580	1.20	0.8849	1.70	0.9554	2.20	0.98610	2.70	0.99653
0.21	0.5832	0.71	0.7611	1.21	0.8869	1.71	0.9564	2.21	0.98645	2.71	0.99664
0.22	0.5871	0.72	0.7642	1.22	0.8888	1.72	0.9573	2.22	0.98679	2.72	0.99674
0.23	0.5910	0.73	0.7673	1.23	0.8907	1.73	0.9582	2.23	0.98713	2.73	0.99683
0.24	0.5948	0.74	0.7704	1.24	0.8925	1.74	0.9591	2.24	0.98745	2.74	0.99693
0.25	0.5987	0.75	0.7734	1.25	0.8944	1.75	0.9599	2.25	0.98778	2.75	0.99702
0.26	0.6026	0.76	0.7764	1.26	0.8962	1.76	0.9608	2.26	0.98809	2.76	0.99711
0.27	0.6064	0.77	0.7794	1.27	0.8980	1.77	0.9616	2.27	0.98840	2.77	0.99720
0.28	0.6103	0.78	0.7823	1.28	0.8997	1.78	0.9625	2.28	0.98870	2.78	0.99728
0.29	0.6141	0.79	0.7852	1.29	0.9015	1.79	0.9633	2.29	0.98899	2.79	0.99736
0.30	0.6179	0.80	0.7881	1.30	0.9032	1.80	0.9641	2.30	0.98928	2.80	0.99744
0.31	0.6217	0.81	0.7910	1.31	0.9049	1.81	0.9649	2.31	0.98956	2.81	0.99752
0.32	0.6255	0.82	0.7939	1.32	0.9066	1.82	0.9656	2.32	0.98983	2.82	0.99760
0.33	0.6293	0.83	0.7967	1.33	0.9082	1.83	0.9664	2.33	0.99010	2.83	0.99767
0.34	0.6331	0.84	0.7995	1.34	0.9099	1.84	0.9671	2.34	0.99036	2.84	0.99774
0.35	0.6368	0.85	0.8023	1.35	0.9115	1.85	0.9678	2.35	0.99061	2.85	0.99781
0.36	0.6406	0.86	0.8051	1.36	0.9131	1.86	0.9686	2.36	0.99086	2.86	0.99788
0.37	0.6443	0.87	0.8078	1.37	0.9147	1.87	0.9693	2.37	0.99111	2.87	0.99795
0.38	0.6480	0.88	0.8106	1.38	0.9162	1.88	0.9699	2.38	0.99134	2.88	0.99801
0.39	0.6517	0.89	0.8133	1.39	0.9177	1.89	0.9706	2.39	0.99158	2.89	0.99807
0.40	0.6554	0.90	0.8159	1.40	0.9192	1.90	0.9713	2.40	0.99180	2.90	0.99813
0.41	0.6591	0.91	0.8186	1.41	0.9207	1.91	0.9719	2.41	0.99202	2.91	0.99819
0.42	0.6628	0.92	0.8212	1.42	0.9222	1.92	0.9726	2.42	0.99224	2.92	0.99825
0.43	0.6664	0.93	0.8238	1.43	0.9236	1.93	0.9732	2.43	0.99245	2.93	0.99831
0.44	0.6700	0.94	0.8264	1.44	0.9251	1.94	0.9738	2.44	0.99266	2.94	0.99836
0.45	0.6736	0.95	0.8289	1.45	0.9265	1.95	0.9744	2.45	0.99286	2.95	0.99841
0.46	0.6772	0.96	0.8315	1.46	0.9279	1.96	0.9750	2.46	0.99305	2.96	0.99846
0.47	0.6808	0.97	0.8340	1.47	0.9292	1.97	0.9756	2.47	0.99324	2.97	0.99851
0.48	0.6844	0.98	0.8365	1.48	0.9306	1.98	0.9761	2.48	0.99343	2.98	0.99856
0.49	0.6879	0.99	0.8389	1.49	0.9319	1.99	0.9767	2.49	0.99361	2.99	0.99861

Table 4.2 The standard normal CDF $\Phi(y)$.

Time function	Fourier Transform
$\delta(\tau)$	1
1	$\delta(f)$
$\delta(\tau-\tau_0)$	$e^{-j2\pi f\tau_0}$
$u(\tau)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$
$e^{j2\pi f_0\tau}$	$\delta(f-f_0)$
$\cos 2\pi f_0\tau$	$\frac{1}{2}\delta(f-f_0)+\frac{1}{2}\delta(f+f_0)$
$\sin 2\pi f_0\tau$	$\frac{1}{2j}\delta(f-f_0)-\frac{1}{2j}\delta(f+f_0)$
$ae^{-a\tau}u(\tau)$	$\frac{a}{a+j2\pi f}$
$ae^{-a\lvert\tau\rvert}$	$\frac{2a^2}{a^2+(2\pi f)^2}$
$ae^{-\pi a^2\tau^2}$	$e^{-\pi f^2/a^2}$
$\mathrm{rect}(\tau/T)$	$T\,\mathrm{sinc}(fT)$
$\mathrm{sinc}(2W\tau)$	$\frac{1}{2W}\,\mathrm{rect}(\frac{f}{2W})$

Note that a is a positive constant and that the rectangle and sinc functions are defined as

$$\mathrm{rect}(x) = \begin{cases} 1 & \lvert x\rvert < 1/2, \\ 0 & \text{otherwise}, \end{cases}$$

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}.$$

Table 1 Fourier transform pairs of common signals.

Time function	Fourier Transform
$g(\tau-\tau_0)$	$G(f)e^{-j2\pi f\tau_0}$
$g(\tau)e^{j2\pi f_0\tau}$	$G(f-f_0)$
$g(-\tau)$	$G^*(f)$
$\frac{dg(\tau)}{d\tau}$	$j2\pi fG(f)$
$\int_{-\infty}^{\tau} g(v)\,dv$	$\frac{G(f)}{j2\pi f}+\frac{G(0)}{2}\delta(f)$
$\int_{-\infty}^{\infty} h(v)g(\tau-v)\,dv$	$G(f)H(f)$
$g(t)h(t)$	$\int_{-\infty}^{\infty} H(f')G(f-f')\,df'$

Table 2 Properties of the Fourier transform.

DETACH

Answer sheet — resit exam EE2S31, 20 July 2022

Name:

Study number:

Question 3(d)

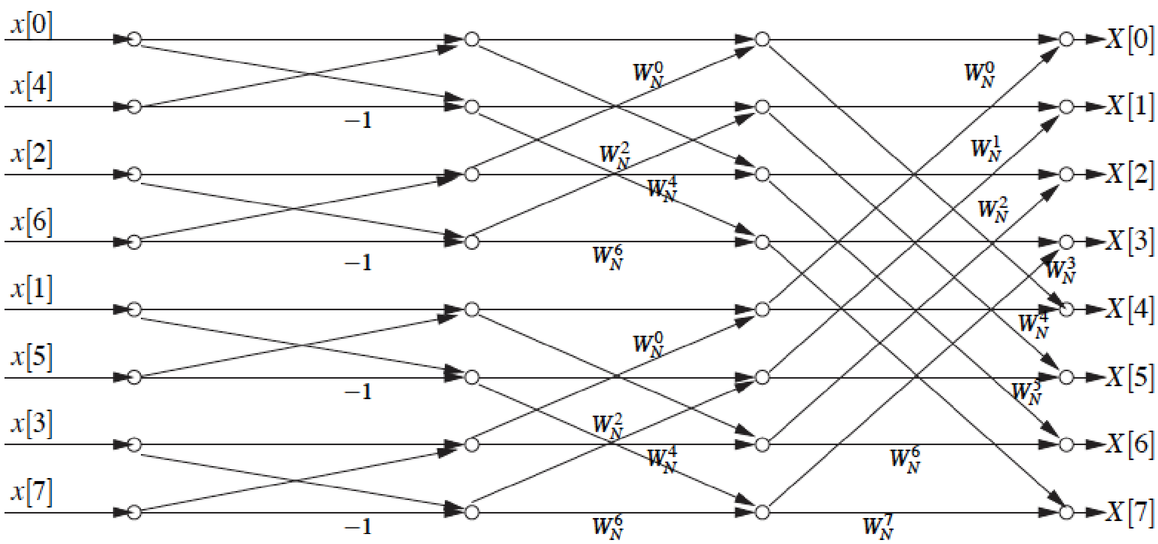


Figure 1. 8-point decimation-in-time algorithm