Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Circuits and Systems

Partial exam EE2S31 SIGNAL PROCESSING
Part 1: May 28, 2021
Block 1: Stochastic Processes (13:30-14:30)

Open book, strictly timed take-home exam. (Electronic) copies of the book and the course slides allowed. No other tools except a basic pocket calculator permitted.
Upload answers during 14:25–14:40

This block consists of three questions (25 points); more than usual, and this will be taken into account during grading. Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.
Hint: Avoid losing too much time on detailed calculations, write down the general approach first.

Question 1 (9 points)

Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} c\,xy & \text{for} \quad 0 \le x \le 1,\ 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the constant c.

(b) Find $f_X(x)$ and $f_Y(y)$.

(c) Are X and Y independent?

(d) Determine $\mathrm{P}[X + Y \le 1]$.

(e) Determine $\mathrm{E}[X | X + Y \le 1]$.

Question 2 (7 points)

It is known that if X is standard normal distributed, $X \sim \text{Gaussian}(0, 1)$, then $Y = X^2$ is Chi-square distributed with 1 degree of freedom.

Further, it is known that if Y has a Chi-square distribution with n degrees of freedom, the moment generating function (MGF) is given by

$$\phi_Y(s) = \frac{1}{(1-2s)^{n/2}}\,, \qquad \text{ROC: } s < \frac{1}{2}$$

(a) Show that if $X_i \sim \text{Gaussian}(0, 1)$ (all independent), then $Y = \sum_1^n X_i^2$ has a Chi-squared distribution with n degrees of freedom.

(b) Use the MGF to prove that

$$\mathrm{E}[Y] = n, \qquad \mathrm{var}[Y] = 2n\,.$$

Suppose now we have n iid random variables $X_i \sim \text{Gaussian}(0, \sigma)$, and we try to estimate the variance σ^2 using

$$S_n = \frac{1}{n} \sum_{i=1}^{n} X_i^2$$

(c) What is $\mathrm{E}[S_n]$ and $\mathrm{var}[S_n]$?

Is the estimate S_n unbiased? Is it consistent?

(d) Use the central limit theorem to estimate how many samples n are at least needed such that

$$\mathrm{P}[\,|S_n - \sigma^2| > 0.1\sigma^2] \; < \; 0.01$$

Note: You will need to use table 4.1/4.2 on p. 129/130.

Question 3 (9 points)

In a BPSK communication system, a source wishes to communicate a random bit X to a receiver. The possible bits $X = 1$ and $X = -1$ are equally likely. In this system, the source transmits X multiple times. In the ith transmission, the receiver observes $Y_i = X + N_i$. After n transmissions of X, the receiver has observed $\boldsymbol{Y} = \boldsymbol{y} = [y_1, \cdots, y_n]^T$.

Assume the noise N_i are iid Gaussian(0,1) random variables, independent of X.

Let $\hat{X}_{\mathrm{ML}}(\boldsymbol{y})$ be the maximum likelihood (ML) estimate of X based on the observation $\boldsymbol{Y} = \boldsymbol{y}$.

(a) Show that

$$f_{\boldsymbol{Y}|X}(\boldsymbol{y}|x) = c\, e^{-\frac{1}{2}\sum_{i=1}^{n}(y_i - x)^2},$$

where c is some constant.

(b) What is $\hat{X}_{\mathrm{ML}}(\boldsymbol{y})$?

(c) Is knowledge that $X \in \{1, -1\}$ used by the ML estimator? Does the noise variance play a role?

(d) Compute e_{ML}, the mean square error of the ML estimate.

Let $\hat{X}_L(\boldsymbol{y}) = \boldsymbol{a}^T\boldsymbol{y} + b$ be the linear minimum mean square error (LMMSE) estimate of X.

(e) Find the LMMSE estimate $\hat{X}_L(\boldsymbol{y})$.

(f) What is e_L, the mean square error of the optimum linear estimate.

Hint: for (e), you may want to exploit Woodbury's Identity,

$$(\boldsymbol{I} + \boldsymbol{u}\boldsymbol{u}^T)^{-1} \;=\; \boldsymbol{I} - \frac{\boldsymbol{u}\boldsymbol{u}^T}{1 + \boldsymbol{u}^T\boldsymbol{u}}$$

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Partial exam EE2S31 SIGNAL PROCESSING
Part 1: May 28th 2021
Block 2: Digital Signal Processing (14:50-15:50)

Open book, strictly timed take-home exam. (Electronic) copies of the book and the course slides allowed. No other tools except a basic pocket calculator permitted. Upload answers during 15:45-16:00.

This block consists of three questions (25 points), more than usual. This will be taken into account during grading. Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. No points will be awarded for results without a derivation. Write your name and student number on each sheet.

Question 4 (11 points)

Let us consider the bicycle wheel on Figure 1. We are taking a video of this wheel at a rate of 24 frames per second. Let us assume that we keep the position of the wheel within the video frame steady.

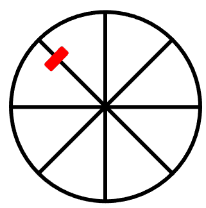

Figure 1

(2 p) (a) What is the maximum angular speed of the wheel that the camera can capture truthfully?

(2 p) (b) At what speed will it appear on the video as if the wheel is standing still? Explain in your own words what is the reason why the wheel appears to stand still!

(2 p) (c) What would happen if we removed the red reflector light? How does your answer to (a) and (b) change?

(3 p) (d) Let's consider now that the wheel is moving at a constant speed and we are capturing the video for infinitely long. The series of values captured by a certain pixel are [1 0 1 0 1 0 ...]. What is the 8-point DFT of this series?

(2 p) (e) In practice, we cannot continue taking the video forever. Let's consider now that we are capturing only 7 frames. Is the 8-point DFT of this series the same as the 8-point DFT of the above infinite series? Why? What about the 8-point DFT of the first 6 frames?

Question 5 (7 points)

Given a real analog signal with a spectrum shown in Figure 2a. We want to sample the signal with 20Hz. We know that the signal contains noise above 5Hz.

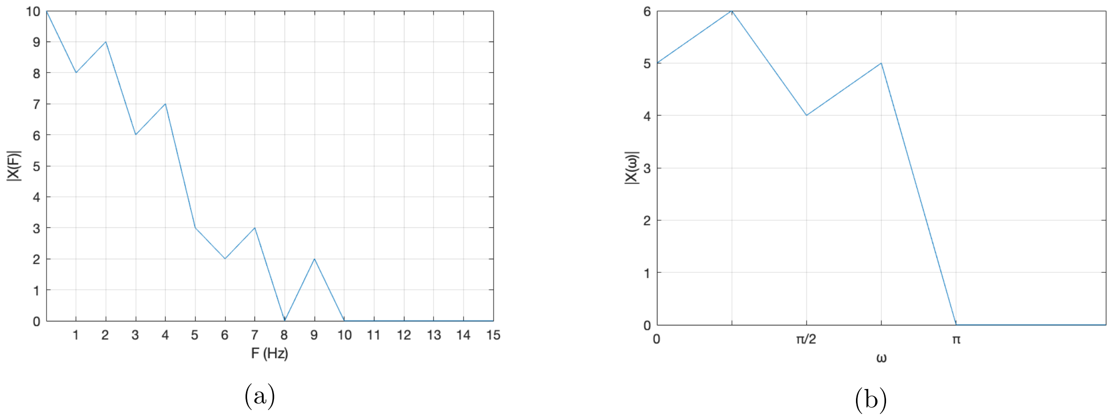

Figure 2

(2 p) (a) Design (sketch) the cheapest possible (non-ideal) antialiasing filter with a linear transition band that preserves the noiseless part of signal.

(2 p) (b) Sketch the spectum of the signal after filtering and sampling! Make sure to correctly indicate the magnitude and frequency values as well as the labels of the axes (pay attention to correct sketching in part (a) too)!

(1 p) (c) Let's assume that after further digital processing, the spectrum of our digital signal is as depicted on Figure 2b. Let's represent this spectrum using an 8-point DFT. What are the values of the DFT coefficients $Y[k]$?

(1 p) (d) Let's further filter the signal with a system with frequency response $H[k]$ = [1 0.9 0.8 0.7 0.6 0.7 0.8 0.9]. What is the DFT of the resulting signal?

(1 p) (e) After filtering using the system $H[k]$ and taking the inverse DFT of the filtered signal, we notice that the first values of the sequence are non-zeros, despite the fact that the original sequence (that corresponds to the spectrum in Figure 2b) are zeros. How do you explain this?

Question 6 (7 points)

(1 p) (a) We are sampling a slowly varying signal with $F_s = 1.05$ Hz sampling rate. How long do we need to sample in order to obtain $N = 21$ sample values?

(1 p) (b) We would like to analyse the spectrum of the acquired sequence. What will be the limit of the frequency resolution?

(3 p) (c) We want to compute the DFT of our sequence using FFT. Give 2 alternative FFT-based algorithms that we could use. Outline the major steps of each algorithm!

(2 p) (d) Which option has lower computational complexity? I.e. how many multiplications do we need to perform for each of the two algorithms?