Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Circuits and Systems

Total points on this exam: 49 pts. The grading was such that 37 points results in grade 10.

Partial exam EE2S31 SIGNAL PROCESSING
Part 2: July 1, 2021
Block 1: Stochastic Processes (13:30-14:30)

Open book, strictly timed take-home exam. (Electronic) copies of the book and the course slides allowed. No other tools except a basic pocket calculator permitted. Upload answers during 14:25–14:40

This block consists of three questions (25 points); more than usual, and this will be taken into account during grading. Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.
Hint: Avoid losing too much time on detailed calculations, write down the general approach first.

Question 1 (7 points)

Let $X(t) = A\cos(\Omega_0 t)$ be a random process, where $A \in \{-1, +1\}$ with equal probabilities, and Ω_0 is a given frequency.

(a) Draw two different realizations of $X(t)$.

(b) What type of random process is $X(t)$? [Think of continuous value/discrete value; continuous-time/discrete time.]

(c) Compute the probability mass function (PMF) $P_{X(t)}(x)$.

(d) Compute $\mathrm{E}[X(t)]$.

(e) Compute $R_X(t, \tau)$.

(f) Is $X(t)$ stationary? Is it WSS?

Solution

(a) 1 pnt (There are only 2 possibilities, one for $A = 1$, the other for $A = -1$)

(b) 1 pnt This is a discrete value continuous-time random process. (Therefore, $X(t)$ is described by a PMF.)

(c) 1 pnt

$$P_{X(t)}(x) = \begin{cases} \frac{1}{2} & x = \cos(\Omega_0 t) \\ \frac{1}{2} & x = -\cos(\Omega_0 t) \\ 0 & \text{otherwise} \end{cases}$$

(d) 1 pnt $\mathrm{E}[X(t)] = \mathrm{E}[A]\cos(\Omega_0 t) = 0.$

(e) 2 pnt Note that $\mathrm{E}[A^2] = 1$. Then

$$R_X(t, \tau) = \mathrm{E}[A\cos(\Omega_0 t) A\cos(\Omega_0(t+\tau))] = \mathrm{E}[A^2]\cos(\Omega_0 t)\cos(\Omega_0(t+\tau)) = \frac{1}{2}\cos(\Omega_0\tau) + \frac{1}{2}\cos(2\Omega_0 t + \Omega_0\tau)$$

(f) 1 pnt Not stationary because $P_{X(t)}(x) \neq P_{X(t+\tau)}(x)$.
Not WSS because $R_X(t, \tau)$ depends on t.

Question 2 (9 points)

Let X_n be an independent identically distributed (iid) random sequence with mean 2 and variance 3, and consider $Y_n = \frac{1}{2}(X_n + X_{n-1})$.

(a) Compute $\mathrm{E}[Y_n]$.

(b) Compute $\mathrm{var}[Y_n]$.

(c) Compute $R_X[k]$.

(d) Compute $R_{XY}[n,k]$ and $R_Y[n,k]$.

(e) Compute the average power of Y_n.

(f) Is Y_n iid? Is it WSS? Is it jointly WSS with X_n?

(g) If X_n is Gaussian, is Y_n Gaussian?

Solution

(a) 1 pnt Use independence of X_n and X_{n-1}: $\mathrm{E}[Y_n] = \frac{1}{2}(\mathrm{E}[X_n] + \mathrm{E}[X_{n-1}) = 2$.

(b) 1 pnt Use independence of X_n and X_{n-1}: $\mathrm{var}[Y_n] = \frac{1}{4}(\mathrm{var}(X_n) + \mathrm{var}(X_{n-1})) = \frac{3}{2}$.

(c) 1 pnt The extended derivation is, using iid,

$$R_X[k] = \mathrm{E}[X_n X_{n+k}] = \begin{cases} \mathrm{E}[X_n^2] & k=0 \\ \mathrm{E}[X_n]\mathrm{E}[X_{n+k}] & k \neq 0 \end{cases} = \begin{cases} \mu_X^2 + \mathrm{var}[X_n] & k=0 \\ \mu_X^2 & k \neq 0 \end{cases} = \begin{cases} 4+3 & k=0 \\ 4 & k \neq 0 \end{cases}$$

Write this in one expression as $R_X[k] = 4 + 3\delta[k]$.

(d) 3 pnt

$$\begin{aligned} R_{XY}[n,k] &= \mathrm{E}[X_n Y_{n+k}] \\ &= \frac{1}{2}\mathrm{E}[X_n(X_{n+k} + X_{n+k-1})] \\ &= \frac{1}{2}(R_X[k] + R_X[k-1]) \\ &= 4 + \frac{3}{2}\delta[k] + \frac{3}{2}\delta[k-1] \end{aligned}$$

$$\begin{aligned} R_Y[n,k] &= \mathrm{E}[Y_n Y_{n+k}] \\ &= \frac{1}{4}\mathrm{E}[(X_n + X_{n-1})(X_{n+k} + X_{n+k-1})] \\ &= \frac{1}{4}(\mathrm{E}[X_n X_{n+k}] + \mathrm{E}[X_n X_{n+k-1}] + \mathrm{E}[X_{n-1} X_{n+k}] + \mathrm{E}[X_{n-1} X_{n+k-1}]) \\ &= \frac{1}{4}(2R_X[k] + R_X[k-1] + R_X[k+1]) \\ &= 4 + \frac{3}{2}\delta[k] + \frac{3}{4}\delta[k-1] + \frac{3}{4}\delta[k+1] \end{aligned}$$

Alternatively, use the convolution equations.

(e) 1 pnt $\mathrm{E}[Y_n^2] = R_Y[0] = 4 + \frac{3}{2} = 5.5$

(f) 1.5 pnt Not iid because Y_n is not independent of Y_{n-1} (they both depend on X_{n-1}). This is also seen from $R_Y[k]$ or, more clearly, from the auto-covariance sequence $C_Y[k] = R_Y[k] - \mu_Y^2 = \frac{3}{2}\delta[k] + \frac{3}{4}\delta[k-1] + \frac{3}{4}\delta[k+1]$: for an iid process we would only have a term with $\delta[k]$.

WSS because $\mathrm{E}[Y_n]$ does not depend on n and $R_Y[n,k]$ does not depend on n.

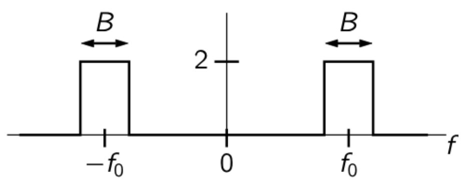
Jointly WSS because both X_n and Y_n are WSS, and $R_{XY}[n,k]$ does not depend on n.

(g) 0.5 pnt Yes, Y_n is also Gaussian distributed, because it is a linear combination of Gaussian variables.

Question 3 (9 points)

The power spectral density $S_X(f)$ of a random process $X(t)$ is given by

$$S_X(f) = \begin{cases} 2 & |f \pm f_0| \le \frac{B}{2} \\ 0 & \text{otherwise} \end{cases}$$



(a) Compute the average power of $X(t)$.

(b) Determine the autocorrelation function $R_X(\tau)$.

Hint: You may need to use Supplement table 1, 2, p. 29/30.

(c) $X(t)$ can be generated by passing white noise through a filter. Assume the noise power spectral density of the input is 1 W/Hz. Specify the filter transfer function $H(f)$.

(d) Let $Y(t) = X(t-5)$. Determine $S_Y(f)$.

Let $Z(t) = 2\,X(t) + N(t)$, where $N(t)$ is independent white noise with power spectral density N_0.

(e) Determine $S_Z(f)$.

(f) Determine $S_{XZ}(f)$ and $S_{NZ}(f)$.

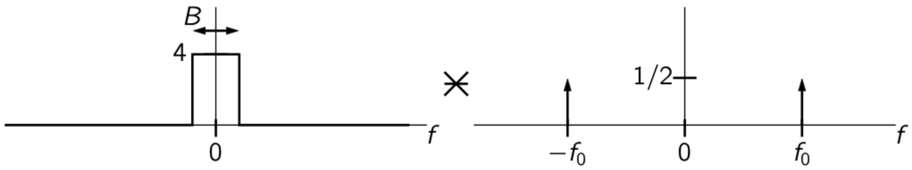
Solution

(a) 1 pnt The average power is the area in the figure:

$$\mathrm{E}[X^2(t)] = R_X(0) = \int_{-\infty}^{\infty} S_X(f)\mathrm{d}f = 4B$$

(b) 3 pnt First recognize that $S_X(f)$ is the convolution of a baseband lowpass filter with two delta pulses in frequency:

$$S_X(f) = S_B(f) * C(f)$$



The autocorrelation function $R_X(\tau)$ is the inverse Fourier transform of $S_X(f)$, hence (see table)

$$R_X(\tau) = R_B(\tau)\, c(\tau)$$

(In Signals & Systems, you learned that there was a factor 2π, however, it disappears because we used f here and not ω.)

Next use the table:

$$\begin{array}{lcl} \text{sinc}(2W\tau) & \leftrightarrow & \frac{1}{2W}\text{rect}\left(\frac{f}{2W}\right) \\ \cos(2\pi f_0 \tau) & \leftrightarrow & \frac{1}{2}\big(\delta(f-f_0)+\delta(f+f_0)\big) \end{array}$$

Note that $B = 2W$. Altogether, this gives

$$R_X(\tau) = 4B\,\text{sinc}(B\tau)\,\cos(2\pi f_0 \tau)$$

(You can check the scale by evaluating $R_X(0) = 4B$, and compare to question (a).)

(c) 1 pnt The filter $H(f)$ needs to satisfy $|H(f)|^2 = S_X(f)$. Hence, it is a bandpass filter,

$$H(f) = \begin{cases} \sqrt{2} & |f \pm f_0| \le \frac{B}{2} \\ 0 & \text{otherwise} \end{cases}$$

where in fact the phase is arbitrary.

(d) 1 pnt The delay in time domain corresponds to a phase shift in frequency domain. This is a filter $G(f)$ with $|G(f)|^2 = 1$. Since $S_Y(f) = |G(f)|^2 S_X(f)$, we have $S_Y(f) = S_X(f)$: the same.

(e) 1.5 pnt The power spectral density of the noise is $S_N(f) = N_0$ (a constant).

Then, since the noise is independent,

$$S_Z(f) = 4S_X(f) + S_N(f) = \begin{cases} 8 + N_0 & |f \pm f_0| \le \frac{B}{2} \\ N_0 & \text{otherwise} \end{cases}$$

(f) 1.5 pnt

$$R_{XZ}(\tau) = \text{E}[X(t)Z(t+\tau)] = \text{E}[X(t)(2X(t+\tau)+N(t+\tau)] = 2\text{E}[X(t)(X(t+\tau)] = 2R_X(\tau)$$

Therefore: $S_{XZ}(f) = 2S_X(f)$. Similarly, $S_{NZ}(f) = S_N(f) = N_0$.

Delft University of Technology
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Section Circuits and Systems

Partial exam EE2S31 SIGNAL PROCESSING
Part 2: July 1st 2021
Block 1: Digital Signal Processing (14:55-15:55)

Open book, strictly timed take-home exam. (Electronic) copies of the book and the course slides allowed. No other tools except a basic pocket calculator permitted. Upload answers during 15:50-16:05.

This block consists of three questions (24 points), more than usual. This will be taken into account during grading. Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. No points will be awarded for results without a derivation. Write your name and student number on each sheet.

Question 4 (11 points)

Let us consider a first-order IIR filter with impulse response

$$h(n) = \frac{1}{4}\left(\frac{1}{2}\right)^{n} u(n) + \frac{1}{3}\left(\frac{1}{2}\right)^{(n-1)} u(n-1)$$

A Direct Form I realization of the filter is shown in Figure 1.

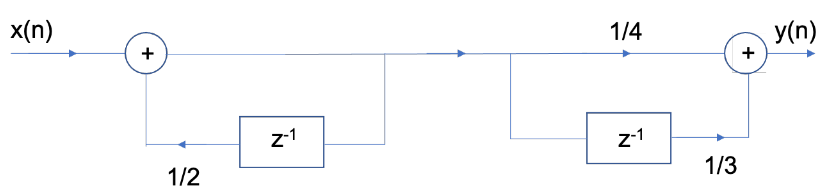

Figure 1

The outputs of the multipliers in this system are quantized using a midtread quantizer and a sign-magnitude coding scheme with 3 bits plus the sign bit. The quantizer can encode values between $(-1, 1)$.

We model the effect of quantization as an additive noise source $e(n)$, and we assume that $e(n)$ is an uncorrelated wide-sense stationary process that is uniformly distributed.

(2 p) (a) What is the variance of the quantization noise of this particular quantizer?

(3 p) (b) Now let us consider the quantization noise at the output of the filter. Compute the variance of the quantization noise at the output of the filter!

(3 p) (c) Give an alternative (Direct Form I) realization of the filter! Is this implementation better or worse in terms of quantization noise power?

(3 p) (d) What is binary code (using the specified quantizer) of the 2nd output sample of the filter in respone to the sequence $[\frac{3}{4}, 0, 0, 0, ...]$, taking into account quantization effects?

Solution

(a) $\sigma_E^2 = \frac{\Delta^2}{12}$ [1 p], where Δ is the step size of the quantizer, that is $\Delta = \frac{R}{2^{b+1}} = \frac{2}{2^{3+1}} = 0.125$ with R the range of the quantizer and b is the number of bits. So, $\sigma_E^2 = \frac{0.125^2}{12} = 0.0013$ [1 p].

(b) For a given quantizatin noise source $\sigma_Q^2 = \sigma_E^2 \sum_{n=-\infty}^{\infty} |h(n)|^2$, where $h(n)$ is the impulse response of part of the system that the noise passes through. [1 p]. The total noise variance is the sum of the output variance of all contributing noise sources. In our case, the first noise source $e1$ (of multiplier $a_1 = \frac{1}{2}$) passes through the whole system, while the other two ($e1$ and $e2$, corresponding to $b_0 = \frac{1}{4}$ and $b_1 = \frac{1}{3}$) appear directly at the output. Therefore,

$$\sigma_{Qtotal}^2 = \sigma_{E1}^2 \sum_{n=-\infty}^{\infty} |h(n)|^2 + \sigma_{E2}^2 + \sigma_{E3}^2 [1p]$$

Let us write the impulse respone of the system in terms of the variables for the clarity of the derivation: $h(n) = b_0 a_1^n u(n) + b_1 a_1^{(n-1)} u(n-1)$
For $m = 0$, the impulse response is $h(0) = b_0$.
For $m \geq 1$, sample m of the impulse response can be written as $h(m) = a_1 b_0 (a_1)^{m\text{-}1}$ $+ b_1 (a_1)^{m\text{-}1}$. Therefore,

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} |h(n)|^2 &= b_0^2 + \sum_{m=1}^{\infty} (a_1 b_0 a_1^{m\text{-}1} + b_1 a_1^{m\text{-}1})^2 \ + \sum_{\text{n}=0}^{\infty} (a_1 b_0 a_1^n + b_1 a_1^n)^2 \\
&= b_0^2 + \sum_{n=0}^{\infty} a_1^2 b_0^2 a_1^{2n} + b_1^2 a_1^{2n} + a_1 b_0 b_1 a_1^{2n} \\
&= b_0^2 \frac{1}{1 - a_1^2} (a_1^2 b_0^2 + b_1^2 + 2 a_1 b_0 b_1) = 0.34
\end{aligned}$$

Substituting back to the previous equation and using the answer to (a):

$$\sigma_{Qtotal}^2 = 0.0013 \cdot (2 + 0.34) = 0.0030$$

(c) An alternative realization is obtained by switching the order of the sections as shown in Figure 2. [1p]. In this case all errors pass through the second section only.[1p] Therefore,

$$\sigma_{Qtotal}^2 = 3 \frac{\Delta^2}{12} \cdot \frac{1}{1 - a_1^2} = 0.0052$$

Which is worse (larger noise power) than the first realization [1p].

(d) We have established that the quantizer has step size 0.125, i.e. quantization levels at 0, $\pm\frac{1}{8}, \pm\frac{2}{8}, ... \pm \frac{7}{8}$. [1 p].

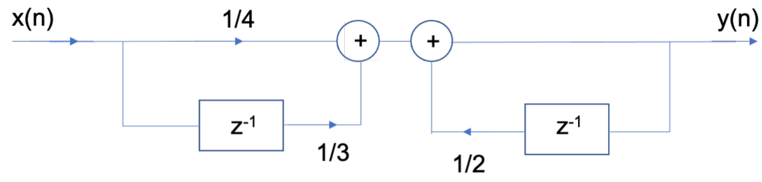

Figure 2

Let us call the output of the first filter section $v(n)$!
The output of the first section is $v(n) = x(n) + \frac{1}{2}v(n-1)$. The final output of the system is $y(n) = \frac{1}{4}v(n) + \frac{1}{3}v(n-1)$. [1 p].
Therefore, $v(0) = \frac{3}{4}$, which is equal to a quantization level, so the quantized value is the same.
$v(1) = 0 + \frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$, which is also a quantization level. Then, $y(1) = \frac{1}{4} \cdot \frac{3}{8} + \frac{1}{3} \cdot \frac{3}{4} = \frac{3}{32} + \frac{1}{4}$. The first term is closest to the quantization level 1/8, i.e. 0.001. The second term is at the quantization level $\frac{2}{8}$, i.e. 0.010. Therefore, the quantized value of the second output sample (i.e. y(1)) equals 0.011. [1 p].

Question 5 (8 points)

Suppose that we want to record a single-channel EEG signal and store it for offline analysis. Our storage capacity is limited, therefore, we will store 100 samples per second. We are interested in analysing the signal in the frequency band 0 − 40Hz, the signal content above 40Hz is of no interest.

We are going to pass the analog EEG signal through the system represented by the block diagram on Figure 3.

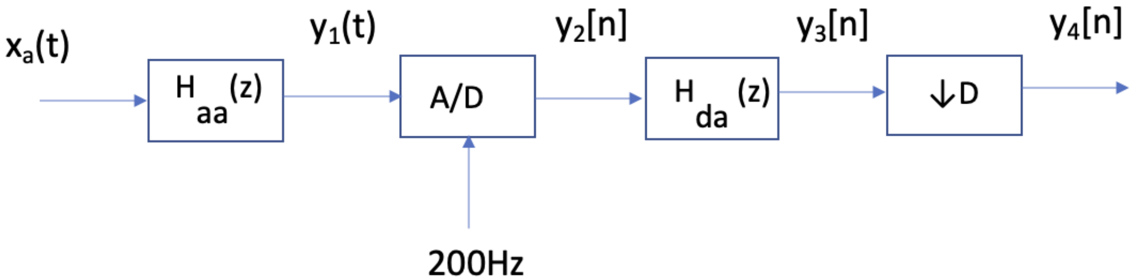

Figure 3

(1 p) (a) What is the purpose of the filters $H_{aa}(z)$ and $H_{da}(z)$?

(1 p) (b) What is the advantage of this system (compared to directly sampling at the desired sampling rate)? Explain!

(2 p) (c) What are the specifications of the filter $H_{aa}(z)$ and $H_{da}(z)$ in terms of pass, stop and transition band?

(1 p) (d) What is the value of D?

(3 p) (e) Assuming that the spectrum of the signal Y_3 is given by the sketch on Figure 4, sketch the spectrum of the signal Y_4! Indicate both physical and normalized frequencies!

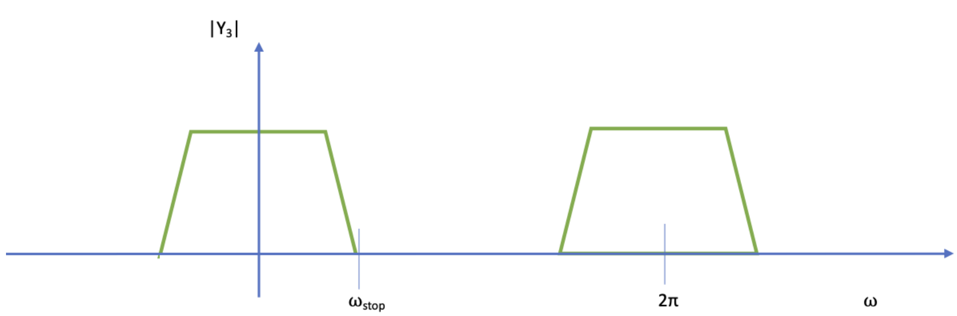

Figure 4

Solution

(a) Anti-aliasing [1 p].

(b) If we first oversample, a less steep anti-aliasing filter is enough in the analog domain. Difficult (steep) filtering can happen in digital domain [1 p].

(c) For $H_{aa}(z)$ the passband is $0 - 40$Hz (to preserve the signal of interest), stopband above 100Hz (for anti-aliasing, due to 200Hz sampling rate of the A/D converter) and therefore transition band between $40 - 100$ Hz. For the digital domain filter H_{da} we need to remove frequencies above 50Hz (as final sampling rate will be 100Hz), so our stopband starts at 50Hz, transition band $40 - 50$Hz, passband the same as for H_{aa} [1 p per filter]

(d) D=2. (Downsampling from 200 to 100 Hz)

(e) The spectrum is given in Figure.[1 p for correct physical frequencies, normalized freequencies and 1 p for correct image (repetitions)] 5.

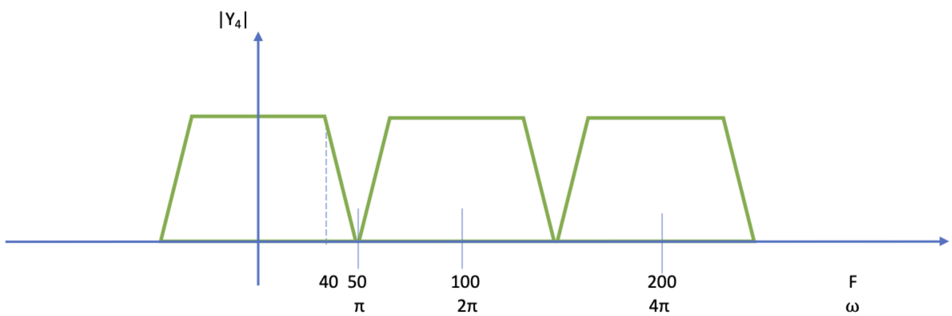

Figure 5

Question 6 (5 points)

Sigma-delta-modulators use prediction, 1-bit quantization and oversampling to achieve high SNR A/D conversion. Let us consider an input signal $x[n]$, with a bandwidth $B = 10\text{kHz}$. The first few values of its autocorrelation sequence R_{xx} are given as $R_{xx}[0] = 100$, $R_{xx}[1] = 42$, $R_{xx}[2] = 35$.

1 p (a) Assume that the signal $y[n]$ is given by:

$$y(n) = x[n] - x[n-1] \tag{1}$$

Is it advantageous to quantize the $y[n]$ instead of $x[n]$?

2 p (b) Suggest a modification to Eq. 1 to improve the quantization scheme! Be as specific as possible!

2 p (c) Recall that the quantization noise power of a first-order SDM can be expressed by

$$\sigma_n^2 \approx \frac{1}{3}\pi^2\sigma_e^2\left(\frac{2B}{F_s}\right)^3 \tag{2}$$

where σ_e^2 is the quantization noise variance F_s is the chosen sampling rate. How should we choose F_s in order to achieve an increase of 18dB in SNR?

Solution

(a) No, because $R_{xx}[1]/R_{xx}[0] < 0.5$. (see book 6.6.1 and 6.6.2 for details)

(b) Use prediction, i.e. $y[n] = x[n] - ax[n-1]$ with $a = R_{xx}[1]/R_{xx}[0] = 42/100$.

(c) According to the formula in Eq. 2, doubling the sampling rate reduces noise power with 9dB ($= 10\log_{10}(2^3)$). So, 4x oversampling will result in 18dB. That is, $F_s = 4 \cdot F_{Nyq} = 4 \cdot 2B = 80\text{kHz}$.