Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Circuits and Systems

Resit exam EE2S31 SIGNAL PROCESSING
July 27, 2021
Block 1: Stochastic Processes (13:30-15:00)

Open book, strictly timed take-home exam. (Electronic) copies of the book and the course slides allowed. No other tools except a basic pocket calculator permitted.
Upload answers during 14:55–15:10

This block consists of three questions (25 points); more than usual, and this will be taken into account during grading. Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.
Hint: Avoid losing too much time on detailed calculations, write down the general approach first.

Question 1 (9 points)

X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} c\,(y-x) & \text{for} \quad 0 \le x \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the constant c.

(b) What is $f_X(x)$ and $f_Y(y)$.

(c) Are X and Y independent?

(d) What is $f_{X|Y}(x|y)$.

(e) What is the blind estimate, $\hat{x}_B$.

(f) What is $\hat{x}_M(y)$, the MMSE estimate of X given $Y = y$.

(g) What is $\hat{x}_{\text{MAP}}(y)$, the maximum a posteriori estimator for X given $Y = y$.

Solution

(a) 1.5 pnt

$$\begin{aligned} \iint f_{X,Y}(x,y)\,\mathrm{d}x\mathrm{d}y &= \int_0^1 \left(\int_0^y c(y-x)\mathrm{d}x \right) \mathrm{d}y \\ &= c \int_0^1 \left[xy - \frac{1}{2}x^2)\mathrm{d}x \right]_0^y \mathrm{d}y \\ &= c \int_0^1 \frac{1}{2} y^2 \mathrm{d}y \\ &= c \int_0^1 \frac{1}{2} y^2 \mathrm{d}y \\ &= \frac{c}{6} \quad = 1\,, \end{aligned}$$

hence $c = 6$.

(b) 2 pnt For $0 \le x \le 1$:

$$\begin{aligned} f_X(x) &= \int f_{X,Y}(x,y)\,\mathrm{d}y \\ &= \int_x^1 c(y-x)\,\mathrm{d}y \\ &= c\left[\frac{1}{2}y^2 - xy\right]_x^1 \\ &= 3x^2 - 6x + 3\,. \end{aligned}$$

Hence,

$$f_X(x) = \begin{cases} 3x^2 - 6x + 3 & 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

For $0 \le y \le 1$:

$$f_Y(y) = \int f_{X,Y}(x,y)\mathrm{d}x = \int_0^y c(y-x)\mathrm{d}x = \left[6(yx - \frac{1}{2}x^2)\right]_0^y = 3y^2$$

Hence,

$$f_Y(y) = \begin{cases} 3y^2 & 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(c) 1 pnt Not independent, because $f_{X,Y}(x,y) \neq f_X(x)\,f_Y(y)$.

(d) 1 pnt

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \dfrac{2}{y} - \dfrac{2}{y^2}\,x & 0 \le x \le y \\ 0 & \text{otherwise.} \end{cases}$$

(e) 1 pnt

$$\begin{aligned} \hat{x}_B &= \mathrm{E}[X] = \int_0^1 x\,f_X(x)\,\mathrm{d}x \\ &= \int_0^1 (3x^3 - 6x^2 + 3x)\,\mathrm{d}x \\ &= \left[\frac{3}{4}x^4 - 2x^3 + \frac{3}{2}x^2\right]_0^1 \\ &= \frac{1}{4} \end{aligned}$$

(f) 1.5 pnt

$$
\begin{aligned}
\hat{x}_{MSE}(y) &= \mathrm{E}[X|y] = \int x\, f_{X|Y}(x|y)\,\mathrm{d}x \\
&= \int_0^y \left(\frac{2}{y}x - \frac{2}{y^2}x^2\right)\mathrm{d}x \\
&= \left[\frac{x^2}{y} - \frac{2}{3}\frac{x^3}{y^2}\right]_0^y \\
&= y - \frac{2}{3}y \quad = \frac{1}{3}y
\end{aligned}
$$

(g) 1 pnt

$$
\begin{aligned}
\hat{x}_{\mathrm{MAP}}(Y=y) &= \arg\max_x f_{X|Y}(x|y) \\
&= \arg\max_{x;\,0\le x\le y} \frac{2}{y} - \frac{2}{y^2}x \\
&= 0
\end{aligned}
$$

since for any given y, the function to be maximized is a line with a negative slope.

Question 2 (7 points)

It is known that if U is standard normal distributed then $Z = U^2$ is Chi-square distributed (with 1 degree of freedom), and that its moment generating function (MGF) is given by

$$
\phi_Z(s) = \frac{1}{\sqrt{1-2s}}\,, \qquad \text{ROC: } s < \tfrac{1}{2}\,.
$$

Let X and Y be independent standard Gaussian variables (i.e., mean 0, variance 1). In this question, we aim to find the MGF of their product, $V = X\,Y$.

(a) Compute the mean and the variance of $X+Y$ and of $X-Y$.

(b) Derive the PDF of $X+Y$ and of $X-Y$.

(c) Show that $X+Y$ is independent of $X-Y$.

(d) Derive that the MGF of $W = (X+Y)^2$ is

$$
\phi_W(s) = \frac{1}{\sqrt{1-4s}}\,.
$$

(e) Derive the MGF of the product $V = X\,Y$.

Hint: First write $X\,Y = \frac{1}{4}(X+Y)^2 - \frac{1}{4}(X-Y)^2$.

Solution

(a) 2 pnt $\mathrm{E}[X+Y] = \mathrm{E}[X] + \mathrm{E}[Y] = 0$, similarly $\mathrm{E}[X-Y] = 0$. Due to independence,

$$\mathrm{var}[X+Y] = \mathrm{var}[X] + \mathrm{var}[Y] = 2\,,$$

and similarly $\mathrm{var}[X-Y] = 2$.

(b) 1 pnt A linear combination of Gaussians is again Gaussian. Hence both are Gaussian with zero mean and variance 2, with the same PDF

$$f(u) = \frac{1}{2\sqrt{\pi}} e^{-u^2/4}\,.$$

(c) 1 pnt Compute the cross-correlation

$$\mathrm{E}[(X+Y)(X-Y)] = \mathrm{E}[X^2 - Y^2] = \mathrm{E}[X^2] - \mathrm{E}[Y^2] = 0\,.$$

Thus, $X+Y$ is uncorrelated with $X-Y$. But for Gaussian variables, uncorrelated means independent (viz. Theorem 5.20).

(d) 1.5 pnt $X+Y$ is a Gaussian with zero mean and variance 2. Thus, we can write $X+Y = \sqrt{2}\,U$, with U a standard normal distributed variable, and $W = (X+Y)^2 = 2U^2$ is a *scaled* Chi-squared variable.

The MGF of $Z = U^2$ is given. The scaling rules are in Theorem 9.5, resulting in

$$\phi_W(s) = \phi_{(X+Y)^2}(s) = \phi_Z(2s) = \frac{1}{\sqrt{1-4s}}$$

(e) 1.5 pnt The scaling rules for $\frac{1}{4}(X+Y)^2$ result in

$$\phi_{1/4\,(X+Y)^2}(s) = \phi_W(\tfrac{1}{4}s) = \tfrac{1}{\sqrt{1-s}}$$

The MGF of $(X-Y)^2$ is the same as that of $(X+Y)^2$. But for the MGF of $-\frac{1}{4}(X-Y)^2$, the scaling by $-\frac{1}{4}$ gives

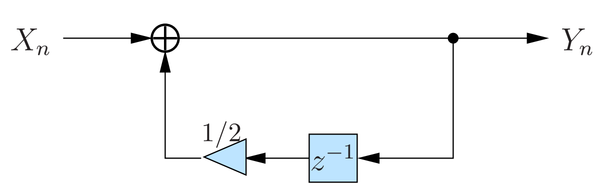
$$\phi_{-1/4\,(X-Y)^2}(s) = \frac{1}{\sqrt{1+s}}$$

The MGF of a sum results in a product of the MGFs, hence

$$\phi_{XY}(s) = \frac{1}{\sqrt{1-s}}\,\frac{1}{\sqrt{1+s}} = \frac{1}{\sqrt{1-s^2}}\,.$$

Question 3 (9 points)

Consider the following system:



The input signal is an iid Gaussian random process X_n, with mean $\mu_X = 2$ and variance $\sigma_X^2 = 3$. The output Y_n satisfies the recursion $Y_n = \frac{1}{2}Y_{n-1} + X_n$.

(a) Determine the autocorrelation sequence of the input, $R_X[k]$, as well as its power spectral density, $S_X(\phi)$.

(b) Compute $\mathrm{E}[Y_n]$.

The autocovariance sequence of the output is

$$C_Y[k] = \frac{4}{3}\left(\frac{1}{2}\right)^{|k|}\sigma_X^2 \,.$$

(c) Compute the autocorrelation sequence $R_Y[k]$ of the output.

(d) What is the average output power?

(e) Determine the power spectral density of the output, $S_Y(\phi)$.

(f) Compute $\mathrm{P}[Y_n > 8]$.

Note: See Table 4.1 or 4.2 (page 129/130) for $\Phi(z)$ or $Q(z)$.

See Table 3 (Suppl. page 38) for Discrete-Time Fourier Transform pairs.

Solution

(a) 2 pnt The input is iid (hence WSS), and

$$R_X[k] = \sigma_X^2\delta[k] + \mu_X^2 = 3\delta[k] + 4\,.$$

The input power spectral density is the DTFT of $R_X[k]$, i.e.,

$$S_X(\phi) = \sigma_X^2 + \mu_X^2\delta(\phi) = 3 + 4\delta(\phi)\,.$$

(b) 2 pnt Using the recursion gives $\mathrm{E}[Y_n] = \frac{1}{2}\mathrm{E}[Y_{n-1}] + \mathrm{E}[X_n]$. Since Y_n is WSS (output of an LTI filter with WSS process as input), $\mathrm{E}[Y_n] = \mathrm{E}[Y_{n-1}] = \mu_Y$, and we find

$$\mu_Y = \frac{1}{2}\mu_Y + \mu_X \qquad \Rightarrow \qquad \mu_Y = 2\mu_X = 4\,.$$

Alternatively, use $\mu_Y = \mu_X \sum_n h[n]$, with $h[n] = (\frac{1}{2})^n$. Then $\sum_n h[n] = \frac{1}{1-1/2} = 2$.

(c) 1 pnt $R_Y[k] = C_Y[k] + \mu_Y^2 = 4\left(\frac{1}{2}\right)^{|k|} + 16$.

(d) 1 pnt $R_Y[0] = 20$.

(e) 1.5 pnt Take the DTFT of $R_Y[k]$. Using Table 3,

$$S_Y(\phi) = \frac{4}{3}\,\frac{1-\frac{1}{4}}{1+\frac{1}{4}-\cos(2\pi\phi)}\,\sigma_X^2 + \mu_Y^2\delta(\phi) = \frac{3}{\frac{5}{4}-\cos(2\pi\phi)} + 16\delta(\phi)$$

Alternatively, use

$$H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}$$

and evaluate $S_Y(\phi) = |H(e^{j2\pi\phi})|^2 S_X(\phi)$:

$$\begin{aligned} S_Y(\phi) &= \frac{1}{1 - \frac{1}{2}e^{-j2\pi\phi}} \frac{1}{1 - \frac{1}{2}e^{j2\pi\phi}} (3 + 4\delta(\phi)) \\ &= \frac{1}{1 + \frac{1}{4} - \frac{1}{2}e^{-j2\pi\phi} - \frac{1}{2}e^{j2\pi\phi}} (3 + 4\delta(\phi)) \\ &= \frac{3}{\frac{5}{4} - \cos(2\pi\phi)} + 16\delta(\phi) \end{aligned}$$

(f) 1.5 pnt We have $\mu_Y = 4$ and $\mathrm{std}(Y_n) = 2$.

$$\mathrm{P}[Y_n > 8] = \mathrm{P}\left[\frac{Y_n - 4}{2} > \frac{8 - 4}{2}\right] = Q(2) = 1 - 0.97725 = 0.0228\,.$$

Resit exam EE2S31 SIGNAL PROCESSING
July 27, 2021
Block 2 (15:25-16:55)

Open book, strictly timed take-home exam. (Electronic) copies of the book and the course slides allowed. No other tools except a basic pocket calculator permitted.
Upload answers during 16:50–17:05

This block consists of three questions (25 points); more than usual, and this will be taken into account during grading. Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.

Question 4 (9 points)

Let us consider the sampling of an amateur radio signal broadcast in the band 3.5-4.0 MHz!

(1 p) (a) Using the graphic in Figure 1 (Figure 6.4.3 from the book), determine the ranges of all possible sampling frequencies that won't result in destructive aliasing! Report your answer by indicating the ranges on the graphic! (You can do this on your computer, or you can sketch this graphic by hand on your answer sheet or use a printed copy of this figure if you have one)

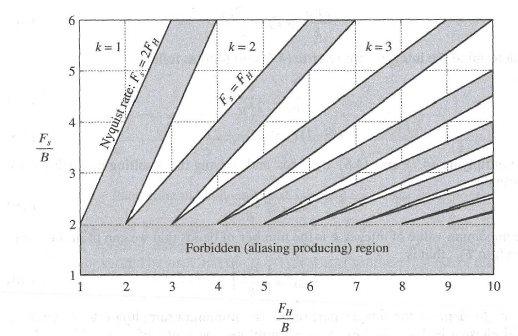


Figure 1

(2 p) (b) Determine the value of a possible sampling frequency which will convert the signal down to baseband, i.e. to 0-0.5 MHz!)

Let us assume that our digital radio receiver samples the radio signal at 1.1025 MHz. After digital demodulation of the baseband signal, we now have a digital audiosignal with a spectrum shown on Figure 2. (Note: there is also noise over the whole spectrum, but this noise is not shown in the figure, only the desired part of the signal.) We want to write this signal onto a CD with sampling rate 44.1 kHz. In order to do that, we first have to downsample the audio signal.

(2 p) (c) Sketch the block diagram of a two-stage downsampler for this task and explain the purpose of each block!

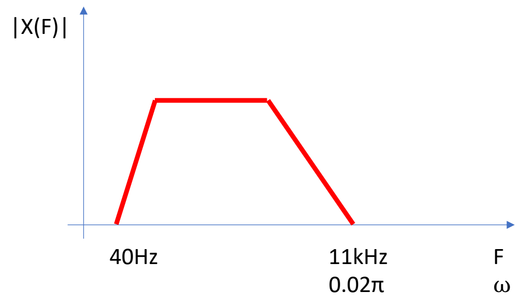


Figure 2

(1 p) (d) What are the decimation factors of each phase in this downsampler?

(2 p) (e) Give the specification of the first filter in your system, in terms of pass/stop/transition band and explain your choice!

(1 p) (f) What is the advantage of this solution over single-stage downsampling?

Solution

(a) $F_H = 4$MHz and $B = 0.5$MHz, therefore, $\frac{F_H}{B} = 8$ .Accordingly, the possible ranges are indicated in red in Fig 3 (only the white areas below the red line)

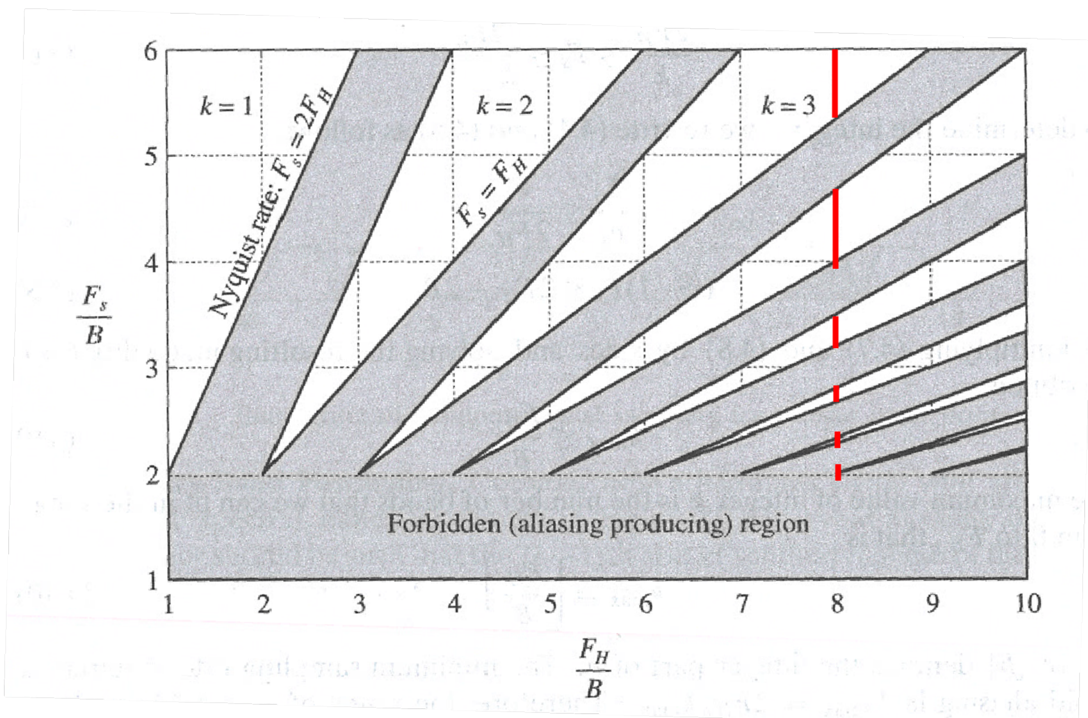


Figure 3

(b) In order to convert to baseband, we need to 'map' 3.5MHz to 0Mhz. Using the formula that relates the digital spectrum to the analog spectrum: $0 = 3.5 - k \cdot F_s$. Taking for example k=2, this gives us $F_s = 3.5/2 = 1.75MHz$. This corresponds to $F_S/B = 1.75/0.5 = 3.5$, which is allowed according to Fig. 3. (it can also be verified by sketching the resulting digital spectrum.) One could choose a different k as well, leading to a different sampling rate.

(c) The block diagram is shown in Fig. 4. It has two decimators, and an anti-aliasing filter before each.

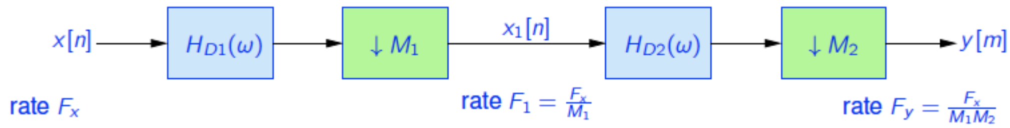


Figure 4

(d) We need a total of $1102.5/44.1 = 25$ factor downsampling. The individual decimators must have an integer factor, so, this is possible to achieve with $M_1 = M_2 = 5$.

(e) After the first downsampling stage, the digital signal has a sampling rate $F_1 = 220.5$kHz, therefore its spectrum will be periodic with copies every $220.5kHz$. In order to prevent aliasing, we need a filter with stopband at 220.5 - 11 kHz (corresponding to the lowest frequency of the first copy of the negative part of the original spectrum). Therefore, the filter specifications are:

- passband: $0 - 11$kHz
- transition band: $11 - (220.5 - 11)$kHz
- stopband: above 220.5kHz.

(1 p) (f) It will allow the use of lower order filters.

Question 5 (8 points)

Consider a digital signal $y[n]$ that is the result of filtering the sequence $x[n] = [\boxed{1}, 0, 2, 0, -1, 0, 3, 0]$ using a digital filter with impulse response $h[n] = [\boxed{3}, 2, 1, 2]$.

(2 p) (a) Compute the values of $y[n]$ in the time domain! Note: Indicate each step of your computations. Without clear intermediate steps, the end result will not be accepted!

(2 p) (b) How can you calculate $y[n]$ in the frequency domain? Write down the steps of the method (no need to make calculations)

(2 p) (c) Determine the 8-point DFT of $x[n]$!

(2 p)(d) Which of the above approaches (time domain or frequency domain) is better in terms of computational complexity (assuming that the DFT matrices are known) for a sequence and a filter of this length, in general?

Solution

(a) According to the formula,

$$
\begin{aligned}
y[n] &= \sum_{k=0}^{M-1} h(k)x(n-k), \text{therefore} \\
h[0]x[n] &= \begin{bmatrix} 3 & 0 & 6 & 0 & -3 & 0 & 9 & 0 \end{bmatrix} \\
h[1]x[n-1] &= \begin{bmatrix} 0 & 2 & 0 & 4 & 0 & -2 & 0 & 6 \end{bmatrix} \\
h[2]x[n-2] &= \begin{bmatrix} 0 & 0 & 1 & 0 & 2 & 0 & -1 & 0 \end{bmatrix} \\
h[3]x[n-3] &= \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 4 & 0 & -2 \end{bmatrix} \text{,such that} \\
y[n] &= \begin{bmatrix} 3 & 2 & 7 & 6 & -1 & 2 & 8 & 4 \end{bmatrix}
\end{aligned}
$$

(b) In the frequency domain, we first need to zero-pad the signals, take their DFT, multiply them and then take the inverse DFT of the result.

(c) We need to multiply the sequence $x[n]$ with the 8-point DFT matrix. However, notice that every second sample of $x[n]$ is 0. Therefore, we can simplify the computation to the following matrix multiplication (taking only every second column of th DFT matrix and only the even samples of the sequence:

$$
\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \\ 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 & 3 \end{bmatrix}^T = \begin{bmatrix} 5 & 2+i & -5 & 2-i & 5 & 2+i & -5 & 2-i \end{bmatrix}^T
$$

d In the time domain we need M^2 multiplications and $M(M-1)$ additions (according to the equation in part (a)), with M=4+8-1=11. That is, 121+110. (Note: in part (a) we omitted a lot of zeros, so in fact we made even less calculations) In the frequency domain, we need to take 11-point DFTs of both sequences (matrix multiplication), which is 2 times M^2 multiplications and $M(M-1)$ additions. We need M multiplications to take the product of the DFTs, and finally M^2 multiplications and $M(M-1)$ additions for the inverse DFT. Therefore, time domain computation for this filtering task is less computationally intensive.

Question 6 (8 points)

Given a multirate conversion system with a block scheme shown in Figure 5 with L=2 and M=5. The sampling rate at the input is 100Hz. The amplitude spectrum $|X(\omega)|$ of the input signal $x[n]$ is depicted in Fig. 6.

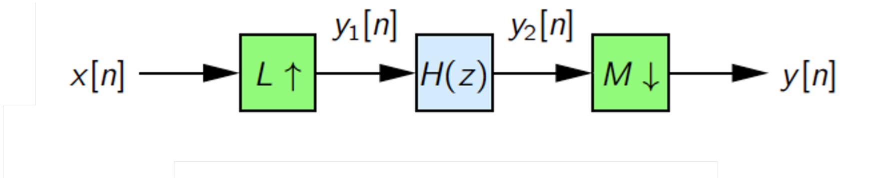


Figure 5

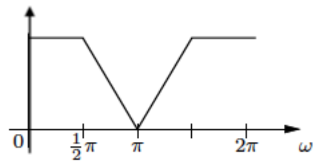


Figure 6

2p (a) Give a formula for $Y_1(\omega)$ in terms of $X(\omega)$ and draw a graphic for the amplitude spectrum $|Y_1(\omega)|$!

Let us consider the implementation of the conversion system shown in Fig. 7.

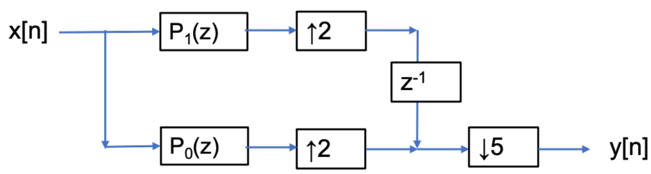


Figure 7

2p (b) What is the role of the filters $P_i(z)$ and how are they related to $H(z)$?

1p (c) At which rate do the filters operate in this implementation?

3p (d) Draw an alternative, more efficient implementation of the multirate conversion system (in terms of the rate at which the filters operate!) At which rate do the filters operate now?

Solution

(a) $Y(\omega) = 2X(\omega)$ The amplitude spectrum is shown below:

(b) $P_i(z)$ are the polyphase representation of the filter $H(z)$, that is $H(z) = P_0(z^2)+z^{-1}P_1(z^2)$.

(c) In the given implementation the filter operates at the same rate as the input, i.e. 100Hz.

(d) An alternative implementation is shown below, where the filters operate at 40Hz. This is obtained from the original block diagram by replacing H(z) with a 5-component polyphase

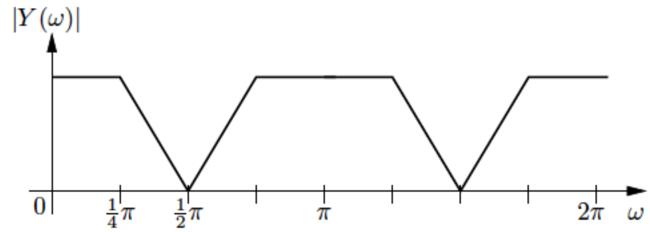


Figure 8

filter and exchanging the order of the filter and the downsampler. One could also work further on the implementation shown on Figure 7, leading to an even more efficient (but more complicated) circuit.

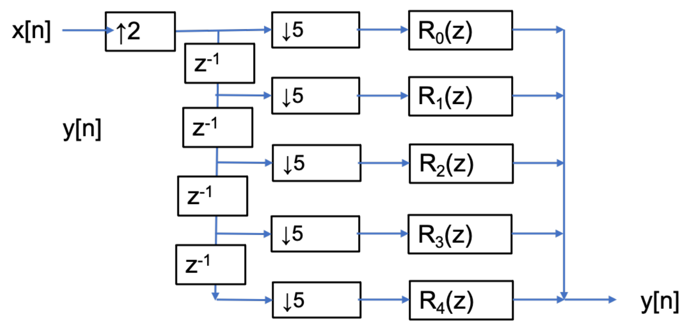


Figure 9