Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Circuits and Systems

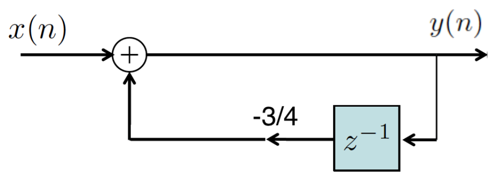
Partial exam EE2S31 SIGNAL PROCESSING
Part 2: June 2nd 2020
Block 1: Digital Signal Processing (13:30-14:30)

Open book, strictly timed take-home exam. (Electronic) copies of the book and the course slides allowed. No other tools except a basic pocket calculator permitted. Upload answers during 14:25-14:35.

This block consists of three questions (22 points), more than usual. This will be taken into account during grading. Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. No points will be awarded for results without a derivation. Write your name and student number on each sheet.

Question 1 (9 points)

Given a recursive system depicted in the figure below.



(a) 1p Write down the difference equation describing the system.

(b) 1p What is the impulse response of the system?

Suppose that the system is implemented with fixed-point arithmetic based on three bits for magnitude and a sign bit (SM representation). Suppose, further, that the quantization that takes place after multiplication rounds the resulting product, such that the maximum quantization error is $\frac{1}{2} \cdot 2^{-b}$.

(c) 1p For this quantizer, make a table with 16 rows and 3 columns, to associate the 4-bit SM representation to the corresponding numerical value. As the first column, show the input range of values that leads to the corresponding quantized output.

(d) 2p Assume an input sequence

$$x[n] = \begin{cases} 0.5 & \text{for } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

- Compute the first 5 samples of the quantized output signal, $y_q[0], \cdots, y_q[4]$.
- What is the range of amplitudes of the dead band of the quantized system?
- From which sample on does the system reach its steady-state output sequence?

(e) 1p Is this steady-state output sequence a constant value or an oscillation? Why?

(f) 1p How can you change the system design (without changing its functionality) in order to decrease the dead band?

(g) 2p Let's consider now an arbitrary input and an additive noise model for the quantization error in the above system. What is the variance of quantization error and the variance of the output noise? Give the numerical values.

Solution

(a) $y(n) = x(n) - \frac{3}{4}y(n-1)$.

(b) $h(n) = (-\frac{3}{4})^n u(n)$.

(c)

y	y_{SM}	y_q
$-\frac{1}{16} \le y < \frac{1}{16}$	0.000	0
$\frac{1}{16} \le y < \frac{3}{16}$	0.001	$\frac{1}{8}$
$\frac{3}{16} \le y < \frac{5}{16}$	0.010	$\frac{1}{4}$
$\frac{5}{16} \le y < \frac{7}{16}$	0.011	$\frac{3}{8}$
$\frac{7}{16} \le y < \frac{9}{16}$	0.100	$\frac{1}{2}$
$\frac{9}{16} \le y < \frac{11}{16}$	0.101	$\frac{5}{8}$
$\frac{11}{16} \le y < \frac{13}{16}$	0.110	$\frac{3}{4}$
$\frac{13}{16} \le y < \frac{15}{16}$	0.111	$\frac{7}{8}$
(unused)	1.000	0
$-\frac{3}{16} \le y < -\frac{1}{16}$	1.001	$-\frac{1}{8}$
$-\frac{5}{16} \le y < -\frac{3}{16}$	1.010	$-\frac{1}{4}$
$-\frac{7}{16} \le y < -\frac{5}{16}$	1.011	$-\frac{3}{8}$
$-\frac{9}{16} \le y < -\frac{7}{16}$	1.100	$-\frac{1}{2}$
$-\frac{11}{16} \le y < -\frac{9}{16}$	1.101	$-\frac{5}{8}$
$-\frac{13}{16} \le y < -\frac{11}{16}$	1.110	$-\frac{3}{4}$
$-\frac{15}{16} \le y < -\frac{13}{16}$	1.111	$-\frac{7}{8}$

(d) The outputs of the system are listed below:

n	$y[n]$	$y_{SM}[n]$	$y_q[n]$	
0	$y[0] = x[0] = \frac{1}{2}$	0.100	$\frac{1}{2}$	
1	$y[1] = x[1] - \frac{3}{4}y_q[0] = 0 - \frac{3}{4} \cdot \frac{1}{2} = -\frac{3}{8}$	1.011	$-\frac{3}{8}$	
2	$y[2] = x[2] - \frac{3}{4}y_q[1] = 0 - \frac{3}{4} \cdot -\frac{3}{8} = \frac{9}{32} = \frac{8+1}{32}$	0.010	$\frac{1}{4}$	rounding down to $\frac{8}{32} = \frac{1}{4}$
3	$y[3] = x[3] - \frac{3}{4}y_q[2] = 0 - \frac{3}{4} \cdot \frac{1}{4} = -\frac{3}{16}$	1.001	$-\frac{1}{8}$	rounding up (alt.: $-\frac{1}{4}$)
4	$y[4] = x[4] - \frac{3}{4}y_q[3] = 0 - \frac{3}{4} \cdot -\frac{1}{8} = \frac{3}{32}$	0.001	$\frac{1}{8}$	rounding down
5	$y[5] = x[5] - \frac{3}{4}y_q[4] = 0 - \frac{3}{4} \cdot \frac{1}{8} = -\frac{3}{32}$	1.001	$-\frac{1}{8}$	rounding down

The quantized system has a dead band with a range of amplitudes between $(-\frac{1}{8}, \frac{1}{8})$. (If the multiplication coefficient was slightly larger or with an alternative rounding rule, it would have been between $(-\frac{1}{4}, \frac{1}{4})$.)

The system is in the steady-state response from $n = 3$ on.

(e) It is an oscillation, because the pole is negative.

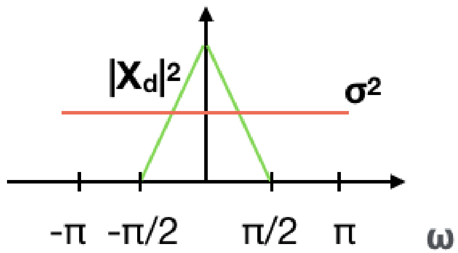
(f) Use more bits. (According to formula 9.6.5 changing the value of the multiplier would also change the dead band, but that would of course change the impulse reponse, i.e., the functionality of the system.)

(g) $\sigma_e^2 = \frac{2^{-2b}}{12} = 0.0013$ and $\sigma_q^2 = \sigma_e^2 \sum\limits_{k=-\infty}^{\infty} h^2(k) = 0.0013 \cdot \frac{1}{1-(-\frac{3}{4})^2} = 0.003$

Question 2 (6 points)

In this question, we consider various methods of improving the signal-to-quantization-noise-ratio in A/D converters: oversampling, noise shaping and differential quantization.

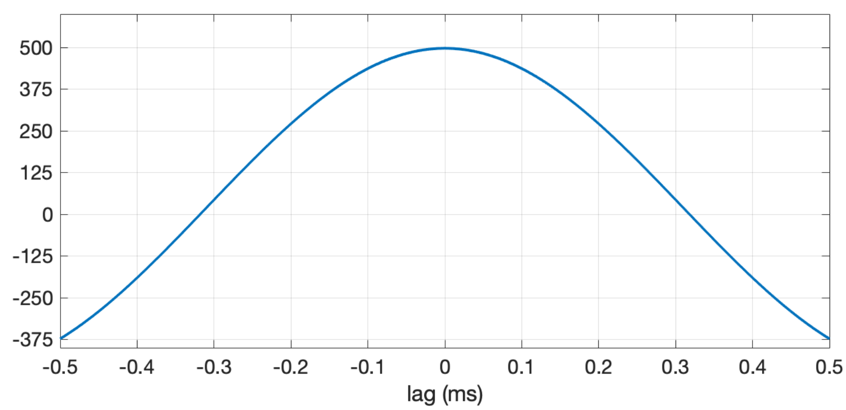
(a) 2p In the figure below, the power spectrum of a digital signal $x_d[n]$ is shown, along with the variance of the quantization noise.



- Explain and illustrate with a similar sketch the effect of oversampling and subsequent downsampling (with a factor of D) on the signal and the noise power spectrum.
- Using a similar sketch, explain the concept of noise shaping.

Next, we consider an analog signal $x_a(t)$ with autocorrelation function $R_{x_a x_a}(\tau)$ shown in the graph below.

Note that the autocorrelation sequence $R_{x_d x_d}[k]$ of the discrete-time signal $x_d[n] = x_a(nT)$ can be expressed as $R_{x_d x_d}[k] = R_{x_a x_a}(kT)$, i.e., it can be considered as the sampled version of the continuous-time autocorrelation function $R_{x_a x_a}(\tau)$ where T is the sampling interval.



(b) 2p Let us consider quantizing

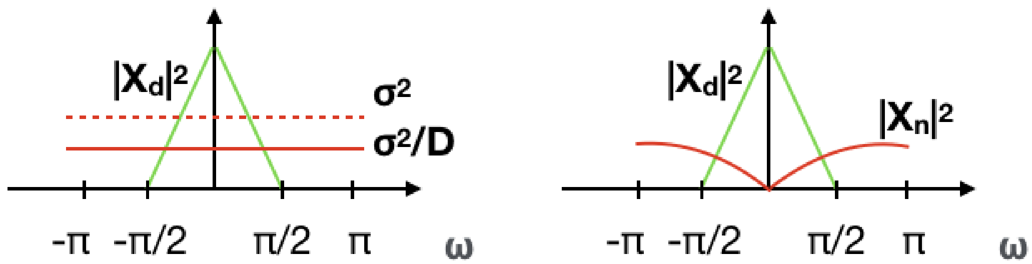
$$d[n] = x_d[n] - x_d[n-1] \tag{1}$$

- How is the variance of the differential signal related to the variance of the original signal? Give the formula!
- What is the minimum sampling rate for which the above differential quantization is beneficial (i.e., the variance of the differential signal is smaller than that of the original)?

(c) 2p Let us now consider a sampling rate of $F_s = 10$ kHz. How would you modify the first order predictor in equation (1) in order to minimize the variance of $d[n]$?

Solution

(a) Oversampling and subsequent downsampling results in a reduction of the noise level by a factor of D, as shown by the sketch on the left. Noise shaping means that the noise is filtered in the band of interest as shown on the sketch on the right. Therefore, noise shaping results in a further increase of SNR.



Note that, in any case, we can filter the signal to exactly its bandwidth ($\pm\frac{\pi}{2}$), which leads to a factor 2 reduction in noise power, but not more. This effect is not meant here.

Note the difference between oversampling and upsampling. For upsampling, we take a sampled signal and insert zeros to increase the rate. That will not be beneficial in this context. For oversampling, we sample at a higher rate, where the desired signal gives redundant (highly correlated) samples while the white noise gives independent noise samples. The LPF used for downsampling will then effectively average the noise, thereby reducing its variance by a factor D.

(b)
- The variance of $d[n]$ is

$$\sigma_d^2 = \mathrm{E}[(x[n] - x[n-1])^2] = 2(R_x[0] - R_x[1])$$

  while the variance of $x_d[n]$ is $\sigma_x^2 = R_x[0]$. Thus, the variance of the differential signal is smaller if

$$\begin{aligned} \sigma_d^2 < \sigma_x^2 \quad &\Leftrightarrow \quad 2(R_x[0] - R_x[1]) < R_x[0] \\ &\Leftrightarrow \quad R_x[0] < 2R_x[1] \\ &\Leftrightarrow \quad \frac{R_x[1]}{R_x[0]} > \frac{1}{2} \end{aligned}$$

  Thus, differential quantization is beneficial in case $R_{x_d x_d}[1]/R_{x_d x_d}[0] > 0.5$.
- As $R_{x_d x_d}[0] = 500$, $R_{x_d x_d}[1]$ should be at least 250. Therefore, $T < 0.2$ ms will work, which coresponds to $F_s = 1/T > 1/0.2\text{ms} = 5$ kHz.

(c) Consider a first order predictor

$$d[n] = x_d[n] - a x_d[n-1]$$

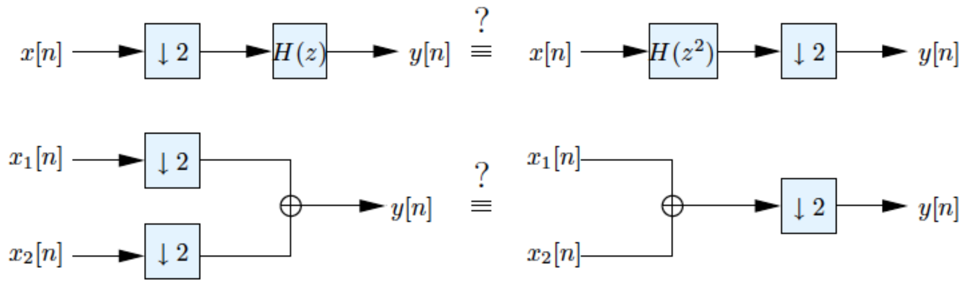
Then, the variance of $d[n]$ is $\sigma_d^2 = (1+a^2)R_x[0] - 2aR_x[1]$. The optimal a (which minimizes σ_d^2) is found as

$$\frac{d}{da}\sigma_d^2 = 2aR_x[0] - 2R_x[1] = 0 \quad \Leftrightarrow \quad a = \frac{R_x[1]}{R_x[0]}$$

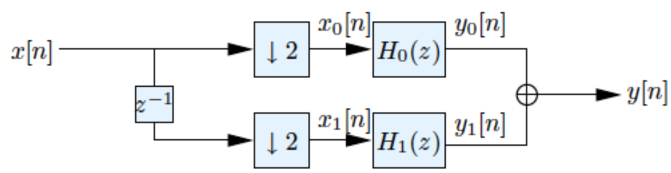
$F_s = 10$ kHz corresponds to $T = 0.1$ ms, for which $R_{x_d x_d}[1] = R_{x_a x_a}(1T) = R_{x_a x_a}(0.1\text{ms}) \approx 437.5$ (around halfway between the grid lines of 375 and 500). Then, $a = \frac{437.5}{500} = 0.875$.

Question 3 (7 points)

(a) 2p Prove the following two identities (potentially, using an example):



(b) 1p Given the following decimation filter:

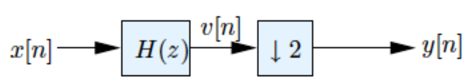


where y_0 and y_1 are generated according to the following difference equations:

$$
\begin{aligned}
y_0[n] &= \frac{1}{4}y_0[n-1] - \frac{1}{3}x_0[n] + \frac{1}{8}x_0[n-1] \\
y_1[n] &= \frac{1}{4}y_1[n-1] + \frac{1}{12}x_1[n]
\end{aligned}
$$

How many multiplications per output sample do you need in this implementation?

(c) 3p The decimation filter can also be implemented using the following system:



where

$$v[n] = av[n-1] + bx[n] + cx[n-1]$$

Determine a, b and c.

(d) 1p How many multiplications per output sample are needed in this implementation?

Solution

(a) For the downsampler in the first identity, it holds that

$$y(n) = x(Dn) \quad \Leftrightarrow \quad Y(z) = \frac{1}{D}\sum_{i=0}^{D-1} X(z^{1/D}W_D^i), \text{ where } W_D = e^{-j2\pi/D}$$

Therefore, the output of the system on the left is

$$Y(z) = H(z)V_1(z) = H(z)\frac{1}{2}\sum_{i=0}^{1} X(z^{1/2}W_2^i) = \frac{1}{2}H(z)\sum_{i=0}^{1} X(z^{1/2}W_2^i)$$

The output of the system on the right can be written as:

$$Y(z) = \frac{1}{2}\sum_{i=0}^{1} V_2(z^{1/2}W_2^i) = \frac{1}{2}\sum_{i=0}^{1} H(zW_2^{2i})X(z^{1/2}W_2^i) = \frac{1}{2}H(z)\sum_{i=0}^{1} X(z^{1/2}W_2^i)$$

where the second equality results from $V_2(z) = H(z^D)X(z)$, and the third equality results from $W_2^{2i} = 1$.

The second identity can be easily shown in the time domain (using arbitrary numbers as example is also possible). Let us consider:

$$x_1[n] = [x_1[0],\, x_1[1],\, x_1[2],\, x_1[3], \cdots] \quad \text{and} \quad x_2[n] = [x_2[0],\, x_2[1],\, x_2[2],\, x_2[3], \cdots].$$

Then, the system on the left gives:

$$y[n] = [x_1[0] + x_2[0],\, x_1[2] + x_2[2],\, x_1[4] + x_2[4],\, \cdots]$$

The system on the right gives (after the adder)

$$v[n] = x_1[n]{+}x_2[n] = [x_1[0]{+}x_2[0],\, x_1[1]{+}x_2[1],\, x_1[2]{+}x_2[2],\, x_1[3]{+}x_2[3],\, x_1[4]{+}x_2[4]\ \cdots],$$

which, after downsampling, gives:

$$y[n] = [x_1[0] + x_2[0],\, x_1[2] + x_2[2],\, x_1[4] + x_2[4],\, \cdots].$$

(b) $y_0[n]$ costs 3 multiplications per sample, $y_1[n]$ costs 2 multiplications per sample, which is a total of 5 per sample.

(c) Using the properties from part (a), the filter from part (b) can be written as

$$\begin{aligned} H(z) &= H_0(z^2) + z^{-1}H_1(z^2) \\ &= \frac{-\frac{1}{3} + \frac{1}{8}z^{-2}}{1 - \frac{1}{4}z^{-2}} + z^{-1}\frac{\frac{1}{12}}{1 - \frac{1}{4}z^{-2}} \\ &= \frac{-\frac{1}{3} + \frac{1}{12}z^{-1} + \frac{1}{8}z^{-2}}{1 - \frac{1}{4}z^{-2}} \\ &= \frac{(1 + \frac{1}{2}z^{-1})(-\frac{1}{3} + \frac{1}{4}z^{-1})}{(1 + \frac{1}{2}z^{-1})(1 - \frac{1}{2}z^{-1})} = \frac{-\frac{1}{3} + \frac{1}{4}z^{-1}}{1 - \frac{1}{2}z^{-1}}. \end{aligned}$$

Therefore, $a = \frac{1}{2}$, $b = -\frac{1}{3}$ and $c = \frac{1}{4}$.

(Obviously, in view of the non-accidental pole/zero cancellation: we would normally start from a given $H(z)$ and then split it into $H_0(z^2)$ and $H_1(z^2)$.)

(d) $v[n]$ costs 3 multiplications per sample. $y[n]$ takes every second sample of $v[n]$. As such, for $y[n + 1]$ we need to calculate $v[2n + 2]$, for which we also have to obtain $v[2n + 1]$ previously. This results in 6 multiplications per sample.

Delft University of Technology
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Section Circuits and Systems

Partial exam EE2S31 SIGNAL PROCESSING
Part 2: July 2, 2020
Block 2: Stochastic Processes (14:35-15:35)

Open book, strictly timed take-home exam. (Electronic) copies of the book and the course slides allowed. No other tools except a basic pocket calculator permitted.
Upload answers during 15:30–15:40

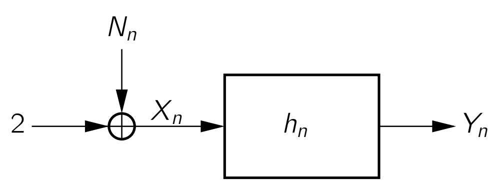
This block consists of three questions (25 points); more than usual, and this will be taken into account during grading. Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.
Hint: Avoid losing too much time on detailed calculations, write down the general approach first.

Question 4 (10 points)

Let the random sequence X_n be a constant 2, perturbed by zero mean i.i.d. noise N_n, with $\mathrm{Var}[N_n] = \sigma^2$.

The random sequence Y_n is obtained by filtering X_n, where the impulse response h_n of the LTI filter is given by

$$h_n = \begin{cases} 1 & n = 0, \\ -\frac{1}{2} & n = 1, \\ 0 & \text{otherwise.} \end{cases}$$



(a) Show that the auto-correlation sequence of X_n is given by

$$R_X[k] = 4 + \sigma^2 \delta[k]\,.$$

(b) Find $\mathrm{E}[Y_n]$.

(c) Find the auto-correlation $R_Y[n,k]$ and the auto-covariance $C_Y[n,k]$.

(d) Is Y_n i.i.d.? Is Y_n wide sense stationary? (Motivate)

(e) Find the cross-correlation $R_{XY}[n,k]$ and cross-covariance $C_{XY}[n,k]$.

(f) Are X_n and Y_n jointly wide sense stationary? (Motivate)

(g) Compute the average power of Y_n.

(h) If, moreover, N_n is Gaussian distributed, then is Y_n Gaussian distributed? (Motivate)

Solution

(a) 1 pnt Since X_n is i.i.d., we know $C_X[k] = \sigma^2\delta[k]$. Then

$$R_X[k] = C_X[k] + \mathrm{E}[X^2] = \sigma^2\delta[k] + 4\,.$$

(b) 1 pnt

$$Y_n = X_n - \frac{1}{2}X_{n-1}$$

$$\mathrm{E}[Y_n] = \mathrm{E}[X_n] - \frac{1}{2}\mathrm{E}[X_{n-1}] = 2 - 1 = 1\,.$$

(c) 2 pnt

$$\begin{aligned}
R_Y[n,k] &= \mathrm{E}[Y_nY_{n+k}] = \mathrm{E}[(X_n - \frac{1}{2}X_{n-1})(X_{n+k} - \frac{1}{2}X_{n+k-1})] \\
&= \mathrm{E}[X_nX_{n+k}] - \frac{1}{2}\mathrm{E}[X_{n-1}X_{n+k}] - \frac{1}{2}\mathrm{E}[X_nX_{n+k-1}] + \frac{1}{4}\mathrm{E}[X_{n-1}X_{n+k-1}] \\
&= 4 + \sigma^2\delta[k] - \frac{1}{2}\left(4 + \sigma^2\delta[k+1]\right) - \frac{1}{2}\left(4 + \sigma^2\delta[k-1]\right) + \frac{1}{4}\left(4 + \sigma^2\delta[k]\right) \\
&= 1 + \sigma^2\left(\frac{5}{4}\delta[k] - \frac{1}{2}\delta[k+1] - \frac{1}{2}\delta[k-1]\right)
\end{aligned}$$

$$C_Y[n,k] = R_Y[n,k] - \mathrm{E}[Y_n]\mathrm{E}[Y_{n+k}] = \sigma^2\left(\frac{5}{4}\delta[k] - \frac{1}{2}\delta[k+1] - \frac{1}{2}\delta[k-1]\right)$$

(d) 1 pnt Not i.i.d.: $C_Y[n,k]$ shows clearly that Y_n is not independent from Y_{n-1}. WSS because $\mathrm{E}[Y_n]$ is independent of n and $C_Y[n,k]$ is independent of n.

(e) 2 pnt

$$\begin{aligned}
R_{XY}[n,k] &= \mathrm{E}[X_nY_{n+k}] = \mathrm{E}[X_n(X_n - \frac{1}{2}X_{n-1})] \\
&= \mathrm{E}[X_nX_n] - \frac{1}{2}\mathrm{E}[X_nX_{n-1}] \\
&= 4 + \sigma^2\delta[k] - \frac{1}{2}\left(4 + \sigma^2\delta[k-1]\right) \\
&= 2 + \sigma^2\left(\delta[k] - \frac{1}{2}\delta[k-1]\right)
\end{aligned}$$

$$C_{XY}[n,k] = \sigma^2\left(\delta[k] - \frac{1}{2}\delta[k-1]\right)$$

(f) 1 pnt Yes, because X_n and Y_n are each WSS, and $C_{XY}[n,k]$ only depends on k.

(g) 1 pnt The average power is $\mathrm{E}[Y_n^2] = R_Y[0] = 1 + \frac{5}{4}\sigma^2$.

(h) 1 pnt Yes because the sum of Gaussian random variables is again a Gaussian random variable.

Question 5 (10 points)

For this question, you may want to use Table 1, printed at the end of the exam.

Assume that $X(t)$ is a white Gaussian noise process with variance $\sigma^2 = 4$. The signal is filtered by an LTI filter with impulse response

$$h(t) = \begin{cases} e^{-5t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The output signal is denoted by $Y(t)$.

(a) Determine $\mathrm{E}[Y(t)]$.

(b) Determine the input power spectral density $S_X(f)$.

(c) Determine the output power spectral density $S_Y(f)$, and sketch a plot of it (indicate values on the axes).

(d) Determine the autocorrelation function of the output, $R_Y(\tau)$.

(e) Determine the average output power, $\mathrm{E}[Y^2(t)]$.

(f) Determine $\mathrm{P}[Y(t) > 0.2]$.

(g) Let $Z(t) = Y(t-3)$. How is the power spectral density of $Z(t)$ related to that of $Y(t)$.

Note: You may express your answer for (f) in terms of $\Phi(z)$ or $Q(z)$.

Solution

(a) 1 pnt $\mathrm{E}[Y(t)] = \mu_X \int_{-\infty}^{\infty} h(t)dt = 0$ as $\mu_X = 0$.

(b) 1 pnt $R_X(\tau) = \sigma^2 \delta(\tau)$ so that $S_X(f) = \sigma^2 = 4$.

(c) 3 pnt Use Table 1, $a = 5$,

$$H(f) = \frac{1}{5 + j2\pi f} = \frac{1}{5 + j\Omega},$$

with $\Omega = 2\pi f$.

$$\begin{aligned} S_Y(f) &= |H(f)|^2 S_X(f) = \frac{\sigma^2}{(5+j\Omega)(5-j\Omega)} \\ &= \frac{\sigma^2}{25 + \Omega^2} \\ &= \frac{4}{25 + 4\pi^2 f^2} \end{aligned}$$

Plot: ...

Cf. a first-order Butterworth response. The -3dB point is at $f = \frac{2\pi}{5}$. The plot is flat at $f = 0$.

(d) 2 pnt The output autocorrelation function is (use Table 1, $a = 5$)

$$R_Y(\tau) = \frac{\sigma^2}{10} e^{-5|\tau|}$$

Therefore, the output power is $\mathrm{E}[Y^2(t)] = R_Y(0) = \frac{\sigma^2}{10} = 0.4$.

(e) 2 pnt We know $\mu_Y = 0$ and $\sigma_Y = \frac{2}{\sqrt{10}}$.

$$\mathrm{P}[Y(t) > 0.2] = \mathrm{P}\left[\frac{Y(t)}{\sigma_Y} > \frac{0.2}{\sigma_Y}\right] = \mathrm{P}\left[\frac{Y(t)}{\sigma_Y} > 0.1\sqrt{10}\right] = Q(0.316) = 0.376.$$

(f) 1 pnt They are the same: $S_Z(f) = S_Y(f)$, because $Z(t) = Y(t) * g(t)$ with $g(t) = \delta(t-3)$; $G(f) = e^{-j3\cdot 2\pi f}$, so that

$$S_Z(f) = |G(f)|^2 S_Y(f) = S_Y(f)\,.$$

Question 6 (5 points)

Suppose that $X(t)$ is a wide sense stationary (WSS) random process with autocorrelation function $R_X(\tau)$.

(a) Is $Y(t) = X(at) + b$ WSS, for arbitrary scalars a, b? If so, specify $R_Y(\tau)$.

(b) Is $Y(t) = X(t-2)$ WSS? If so, specify $R_Y(\tau)$.

(c) Is $Y_n = X(nT)$ a WSS sequence, for $T > 0$? If so, specify $R_Y[k]$.

(d) Suppose that $R_X(\tau) = \delta(\tau) - 1$. Is this a valid autocorrelation function of $X(t)$? (Motivate)

(e) Give an example of a random sequence X_n, that has $R_X[k] = (-1)^k + \delta[k]$ as autocorrelation sequence.

Note: a yes/no answer without motivation will not receive points.

Solution

(a) 1 pnt Yes: Let $\mu_X = \mathrm{E}[X(t)]$ (a constant, because of WSS), then $\mathrm{E}[Y(t)] = \mu_X + b$ is constant, and

$$R_Y(t,\tau) = \mathrm{E}[Y(t)Y(t+\tau)] = \mathrm{E}[(X(at)+b)(X(a(t+\tau))+b)] = R_X(a\tau) + 2b\,\mu_X + b^2 = R_Y(\tau)$$

does not depend on t.

(b) 1 pnt Yes: $\mathrm{E}[Y(t)] = \mu_X$ is constant, and

$$R_Y(t,\tau) = \mathrm{E}[Y(t)Y(t+\tau)] = \mathrm{E}[X(t-2)X(t-2+\tau)] = R_X(\tau)$$

does not depend on t.

(c) 1 pnt Yes; $\mathrm{E}[Y_n] = \mu_X$ is constant, and

$$\mathrm{E}[Y_n Y_{n+k}] = \mathrm{E}[X(nT)X((n+k)T)] = R_X(kT)$$

does not depend on n.

(d) 1 pnt This is in general not straightforward to determine. Clearly it is independent of time t. It satisfies the three conditions (Thm 13.12). However, the corresponding power spectral density would be

$$S_X(f) = 1 - \delta(f)$$

and this is not positive for all f (namely, for $f = 0$ it is $-\infty$). So it is not a valid autocorrelation function.

(e) 1 pnt E.g., $X_n = (-1)^n A + N_k$, where A is a random variable with $\mathrm{E}[A^2] = 1$, and N_k is a white noise sequence (unit variance).

Note: your example must be a *random* sequence, or else you cannot determine $R_X[k]$.

Time function	Fourier Transform
$\delta(\tau)$	1
1	$\delta(f)$
$\delta(\tau-\tau_0)$	$e^{-j2\pi f\tau_0}$
$u(\tau)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$
$e^{j2\pi f_0\tau}$	$\delta(f-f_0)$
$\cos 2\pi f_0\tau$	$\frac{1}{2}\delta(f-f_0)+\frac{1}{2}\delta(f+f_0)$
$\sin 2\pi f_0\tau$	$\frac{1}{2j}\delta(f-f_0)-\frac{1}{2j}\delta(f+f_0)$
$ae^{-a\tau}u(\tau)$	$\frac{a}{a+j2\pi f}$
$ae^{-a\|\tau\|}$	$\frac{2a^2}{a^2+(2\pi f)^2}$
$ae^{-\pi a^2\tau^2}$	$e^{-\pi f^2/a^2}$
$\mathrm{rect}(\tau/T)$	$T\,\mathrm{sinc}(fT)$
$\mathrm{sinc}(2W\tau)$	$\frac{1}{2W}\,\mathrm{rect}(\frac{f}{2W})$

Note that a is a positive constant and that the rectangle and sinc functions are defined as

$$\mathrm{rect}(x)=\begin{cases}1 & |x|<1/2,\\ 0 & \text{otherwise},\end{cases}$$

$$\mathrm{sinc}(x)=\frac{\sin(\pi x)}{\pi x}.$$

Table 1 Fourier transform pairs of common signals.