Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Circuits and Systems

Resit exam EE2S31 SIGNAL PROCESSING
July 31, 2020
Block 1: Stochastic Processes (13:30-15:00)

Open book, strictly timed take-home exam. (Electronic) copies of the book and the course slides allowed. No other tools except a basic pocket calculator permitted.
Upload answers during 14:55–15:10

This block consists of four questions (29 points); more than usual, and this will be taken into account during grading. Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.
Hint: Avoid losing too much time on detailed calculations, write down the general approach first.

Question 1 (8 points)

Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} c & -1 \le x \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the constant c.

(b) Find $f_Y(y)$ and $f_{X|Y}(x|y)$.

(c) Find $\mathrm{E}[Y]$.

(d) Find $\hat{X}_M(Y)$, the minimum mean square error (MMSE) estimator for X given a single sample of Y.

(e) Find $\hat{Y}_{\mathrm{ML}}(X)$, the maximum likelihood estimator for Y given a single sample of X.

Question 2 (7 points)

We flip a coin n times and count how many times we observe "heads"; let $\hat{P}_n$ be the estimated probability of heads (the relative frequency of occurence).

Assume the coin is fair; the true probability of heads is $p = 0.5$.

(a) Find $\mathrm{E}[\hat{P}_n]$ and $\mathrm{Var}[\hat{P}_n]$.

(b) Use the Chebyshev inequality to find an upper bound on the probability that $\hat{P}_{100}$ is within 0.1 of p.

(c) Is $\hat{P}_n$ a consistent estimator of p? (Motivate.)

(d) For large n, we assume that $\hat{P}_n$ can be approximated by a Gaussian distribution. Use this to estimate the probability that $\hat{P}_{100}$ is within 0.1 of p.

Note: See Table 4.1 or 4.2 (page 129/130) for $\Phi(z)$ or $Q(z)$.

Question 3 (8 points)

Let W be an exponentially distributed random variable, with pdf

$$f_W(w) = \begin{cases} e^{-w} & w \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Consider the random process $X(t) = t - W$.

(a) Draw three realizations of $X(t)$.

(b) Determine the CDF $F_W(w)$ and $F_{X(t)}(x)$, and the pdf of $X(t)$.

(c) Determine E$[W]$ and compute the expected value function, $\mu_{X(t)}$.

(d) Determine E$[W^2]$ and compute the autocorrelation function, $C_X(t, \tau)$.

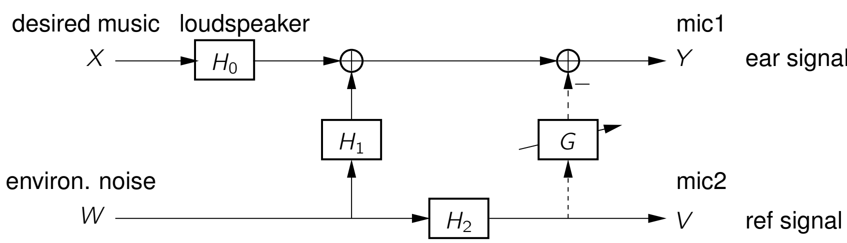
(e) Is $X(t)$ a WSS random process? (Motivate)

(f) Is it an i.i.d. process? (Motivate)

Question 4 (6 points)

In this question, all signals are considered in the frequency domain.

The fundamentals behind a noise canceling headphone are schematically drawn in the figure.



We wish to listen to music $X(f)$ transmitted over a loudspeaker with unknown response $H_0(f)$, but the ear signal $Y(f)$ is disturbed by unknown environment noise $W(f)$, which has been filtered by an unknown channel response $H_1(f)$. We measure the ear signal with microphone 1.

An additional microphone (mic2) also captures the noise signal, but it is filtered by an unknown filter $H_2(f)$.

We wish to design a filter $G(f)$ such that the noise signal on Y is perfectly canceled. **While we design $G(f)$, it is not included in the schematic.**

(a) What is the desired solution for $G(f)$ in terms of $H_1(f)$ and $H_2(f)$?

(b) Show that $H_2^{-1}(f) = |H_2(f)|^{-2} H_2^*(f)$.

$X(f)$ and $W(f)$ are considered to be independent random processes, with power spectral densities $S_X(f)$ and $S_W(f)$, respectively.

(c) Give expressions for $S_Y(f)$, $S_V(f)$ and $S_{YV}(f)$ in terms of $S_X(f)$ and $S_W(f)$.

(d) Which of these (cross) power spectral densities can we observe?

(e) Give an expression for $G(f)$ in terms of observed quantities.

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July 31, 2020
Block 2: Digital Signal Processing (15:00-16:30)

Open book, strictly timed take-home exam. (Electronic) copies of the book and the course slides allowed. No other tools except a basic pocket calculator permitted.
Upload answers during 16:30-16:45.

This block consists of three questions (25 points), more than usual. This will be taken into account during grading. Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. No points will be awarded for results without a derivation. Write your name and student number on each sheet.

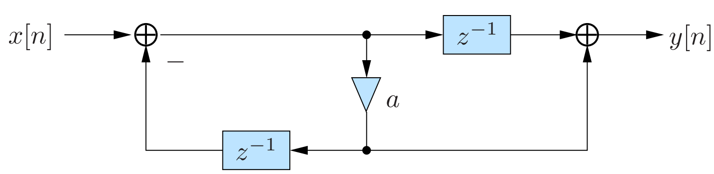
Question 5 (7 points)

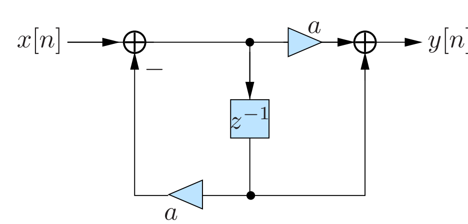
A transfer function $H(z)$ is an allpass function if it satisfies $|H(e^{j\omega})|^2 = 1$. Let $h[n]$ be the corresponding impulse response.

(a) Use a simple argument to show that, for any allpass function, $\sum_n |h[n]|^2 = 1$.

Given two realizations of a first-order allpass function:

$$H(z) = \frac{a + z^{-1}}{1 + az^{-1}}, \qquad |a| < 1$$





In the realization in hardware, each multiplier rounds the resulting product to a finite number of bits, such that the roundoff error is in the interval $[-\frac{\Delta}{2}, \frac{\Delta}{2}]$.

This is modeled by introducing a noise source $e_i[n]$ at the output of each multiplier ($i = 1, 2, 3$). Assume that all roundoff errors are iid random variables with uniform distribution.

(b) What is the variance of the equivalent noise source at the output of each multiplier?

(c) Determine the transfer function $H_i(z)$ of each $e_i[n]$ to the output $y[n]$.

(d) For each realization, determine the variance of the quantization noise at the output $y[n]$. Which realization is preferred?

(e) What are two other aspects based on which we would prefer one realization over the other?

Question 6 (9 points)

(a) Why would we ever want to use a circular convolution?

(b) Compute the circular convolution (of length 5) of the signal $h[n] = [\boxed{1}, 2, 0, 0, 0]$ with $x[n] = [\boxed{1}, 2, 3, 4, 5]$. (Show the steps in the calculation.)

(c) If $h[n]$ has length L samples, and $x[n]$ has M samples, then how can we use a circular convolution to implement a linear (normal) convolution?

We sample an audio signal $x_a(t)$ at 40 kHz, resulting in $x[n] = x_a(nT)$. We sample during 2.5 seconds. Next, we filter the signal by an FIR filter $h[n]$ that has 1024 coefficients.

(d) What is the maximal permissible frequency in the signal $x_a(t)$?

(e) How many multiplications do we need for a direct implementation of the convolution $y = h * x$? (You may ignore edge effects.)

(f) To reduce the number of operations, we implement the convolution by an FFT using the overlap-add technique.

Describe step-by-step how this can be done. Also draw a block diagram of this process.

(g) How many multiplications are needed now?

[Assume that an FFT of order N has a complexity of $N\ \log_2(N)$ multiplications.]

(h) As a side effect of this technique, we can simply produce a time-frequency diagram of $x[n]$. What is the resolution in time and in frequency in this diagram?

Question 7 (9 points)

An audio signal has frequencies until 20 kHz. We are given $x[n]$ in studio quality: the sample rate is 48 kHz. To transfer this to a CD, we need to reduce the sample rate to 44 kHz.

For this, we are using the following process:

$$\underset{\text{rate 48 kHz}}{x[n]} \longrightarrow \boxed{L\uparrow} \xrightarrow{y_1[n]} \boxed{H(z)} \xrightarrow{y_2[n]} \boxed{M\downarrow} \longrightarrow \underset{\text{rate 44 kHz}}{y[n]}$$

(a) Determine suitable values for the upsampling factor L and downsampling factor M.

(b) What is the role of the filter $H(z)$? Give suitable specifications.

(c) Draw schematically spectra of $x[n]$, $y_1[n]$, $y_2[n]$ and $y[n]$. (Indicate values for both ω and F on the frequency axis.)

(d) Is it allowed to swap the upsampler and downsampler?

(e) Suppose that the required filter $H(z)$ is an FIR filter with 264 coefficients. How many multiplications per second are needed to implement the filter in the shown process?

(f) Can this also be done more efficiently? (If so, indicate how, and how much more efficiently.)