Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Circuits and Systems

Partial exam EE2S31 SIGNAL PROCESSING
Part 1: May 27th 2019

Closed book; two sides A4 of handwritten notes permitted

This exam consists of four questions (34 points). Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.

Question 1 (10 points)

(2 p) (a) (1 point): $P(X \geq 1) = \int_1^\infty \frac{1}{\lambda} e^{-\frac{1}{\lambda}x} dx = \left[-e^{-\frac{1}{\lambda}x} \right]_1^\infty = e^{-\frac{1}{\lambda}}$.

(1 point): The conditional pdf $f_{X|X\geq 1}(x)$ is now given by

$$f_{X|X\geq 1}(x) = \begin{cases} \frac{1}{\lambda} e^{-\frac{1}{\lambda}(x-1)} & \text{for} \quad x \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(2 p) (b)

$$\phi_{X|X\geq 1}(s) = \frac{1}{\lambda} e^{\frac{1}{\lambda}} \int_1^\infty e^{-\frac{1}{\lambda}x} e^{xs} dx = \frac{1}{\lambda} e^{\frac{1}{\lambda}} \int_1^\infty e^{x(s-\frac{1}{\lambda})} dx = \frac{\frac{1}{\lambda} e^s}{\frac{1}{\lambda} - s},$$

for $s \leq \frac{1}{\lambda}$ (due to region of convergence of the integral).

(2 p) (c) $E[X|X \geq 1] = \left. \frac{\partial \phi_{X|X\geq 1}(s)}{\partial s} \right|_{s=0} = \left. \frac{\frac{1}{\lambda} e^s}{\frac{1}{\lambda} - s} + \frac{\frac{1}{\lambda} e^s}{(\frac{1}{\lambda} - s)^2} \right|_{s=0} = 1 + \lambda.$

(2 p) (d) $E[\hat{\lambda}] = \frac{\sum_{n=1}^N E[X_n]}{N} = E[X] = \lambda$. The estimator is thus unbiased. $E[X]$ can be calculated using the MGF: $\left. \frac{\partial \phi_X(s)}{\partial s} \right|_{s=0} = \lambda$

(2 p) (e) The variance $var[X_n] = E[X_n^2] - E[X_n]^2$. $E[X^2]$ can be calculated using the MGF: $\left. \frac{\partial^2 \phi_X(s)}{\partial s^2} \right|_{s=0} = 2\lambda^2$. Thus, $var[X_n] = \lambda^2$

An estimator is consistent if

$$\lim_{n\to\infty} P\left[|\hat{\lambda} - \lambda| \geq \epsilon \right] = 0.$$

Based on Chebyshev we now that we can write

$$\lim_{n\to\infty} P\left[|\hat{\lambda} - \lambda| \geq \epsilon \right] \leq \lim_{n\to\infty} \frac{var[\hat{\lambda}]}{\epsilon^2} = \lim_{n\to\infty} \frac{\frac{N\lambda^2}{N^2}}{\epsilon^2} = 0.$$

The estimator is thus consistent.

Question 2 (9 points)

(1p) (a) 12 kHz.

(1p) (b) $H(F) = \begin{cases} 1, & \text{if } |F| \leq 4.5\text{kHz} \\ 0, & \text{otherwise} \end{cases}$

A lowpass filter with a cut-off at 4Hz or a bandpass filter is also an acceptable solution.

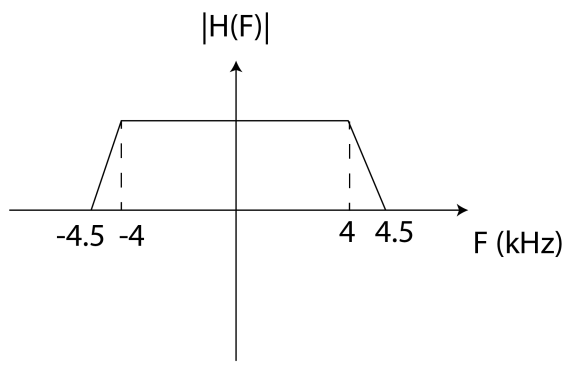
(1p) (c) After filtering, the signal is an integer band positioned passband signal. Therefore, it can be sampled according to the bandwidth B:
$F_s = 2B = 2 \cdot (4 - 3) = 2$ kHz.

(3p) (d) (1 point) The analog signal can be reconstructed using an interpolation function $g(t)$ according to the following formula:

$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a(nT)g(t - nT), \text{ with}$$
$$g(t) = \frac{sin\pi Bt}{\pi Bt} cos2\pi F_c t, \text{ where } B = 2\text{kHz and } F_c = 3.5\text{kHz}$$

(1 point) In practice, we have to use a deterministic and finite interpolation function instead of the sinc function in $g(t)$ above. (1 point) In the frequency domain this corresponds to a non-ideal bandpass filter.

(1p) (e) The non-ideal filter can have a transition band between 4 and 4.5 kHz, e.g. as shown below:



(2p) (f) (1 point) The useful part of the signal is between 3-4kHz. Aliasing must not affect this range, but may affect 4.5-6kHz.
(1 point) Solving the following equations:

$$(k-1) \cdot F_s - 3 \leq 3$$
$$k \cdot F_s - 6 \geq 4$$

gives $F_s = 5$ kHz with $k_{max} = 2$

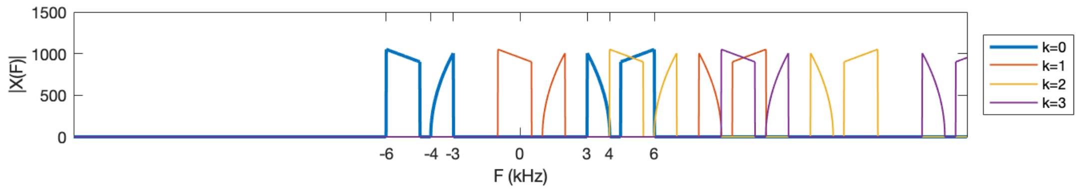


Figure 1. A sketch of the spectrum of the sampled signal can be helpful to write the equations: The spectrum of the analog signal is shown in blue. The periodic copies due to sampling are shown in red, yellow and purple. The highest frequency of the red copy must not be larger than 3kHz, while the lowest frequency of the yellow copy must not be smaller than 4kHz.

Question 3 (7 points)

(2 p) (a) $\int_{y=-1}^{y=0}\int_{x=-1}^{x=y} c(y+x)dxdy = \int_{y=-1}^{y=0}\left[c(yx+\frac{1}{2}x^2)\right]_{-1}^{y} dxdy = \int_{y=-1}^{y=0} c(3/2y^2+y-1/2)dy = -c/2 = 1$. So $c = -2$.

(3 p) (b)

$$f_X(x) = \int_x^0 c(y+x)dy = \left[\frac{1}{2}cy^2 + cxy\right]_x^0 = -\frac{3}{2}cx^2$$

$$f_Y(y) = \int_{-1}^{y} c(y+x)dx = \left[\frac{1}{2}cx^2 + cxy\right]_{-1}^{y} = \frac{3}{2}cy^2 - \frac{1}{2}c + cy.$$

Clearly, X and Y are not independent as $f_{X,Y} \neq f_X f_Y$.

(2 p) (c) $f_{Y|X} = \frac{x+y}{\frac{-3}{2}x^2}$

$E[Y|X] = \int_x^0 \frac{y^2+yx}{-\frac{3}{2}x^2}dy = \frac{5}{9}x$.

Question 4 (8 points)

Sequence #	DFT #
1	5
2	2
3	1
4	8
5	7
6	6
7	4
8	3

Explanation:

- Sequence 1 contains 2 full cycles of a sine, so X(2) and X(20-2) are nonzero, the other frequency samples are 0, just like DFT 5.
- Sequence 2 contains 3 full cycles of a sine, therefore, its DTF is DFT 2.
- Sequence 3 contains 2.5 cycles of a sine. All DFT coefficients are nonzero, the largest values must be X(2), X(3), X(20-2) and X(20-3), that is DFT1.
- Sequence 4 is a single cycle of sine with 10 samples that is zero-padded to 20 samples. Without zero-padding, the 10-point DFT would be all zeros except at X(1) and X(9). Zero-padding gives more samples in between, that are all nonzeros. That is DFT8.
- Sequence 5 is constant, its DFT is an impulse, that is DFT7. It can be shown by solving the expression for DFT, and using the formula for the geometric series (similarly as shown for sequence 8 below).
- Sequence 6 is a constant multiplied with a rectangular window. Therefore, its DFT contains samples of an impulse in the frequency domain, convolved with the Fourier transform of the rectangular window. That is DFT 6.
- Sequence 7 is an imulse, its DFT is a constant, that is DFT4. It can be shown by solving the expression for DFT.
- Sequence 8 is an impulse train. Its Fourier transform is also an impulse train, i.e. DFT3. Alternatively, it can be computed as follows:

$$X(k) = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)nk} = \sum_{r=0}^{4} e^{-j(2\pi/20)\cdot 4\cdot r\cdot k} = \sum_{r=0}^{4} (e^{-j(2\pi/20)\cdot 4\cdot k})^r$$

$$\text{with } a = e^{-j(2\pi/20)\cdot 4\cdot k} \text{ and } \sum_{r=0}^{R-1} a^R = \frac{1-a^R}{1-a}$$

$$X(k) = 4 \text{ if } (2\pi/20 \cdot 4 \cdot k) \text{ is an integer and 0 otherwise.}$$