Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Circuits and Systems

Partial exam EE2S31 SIGNAL PROCESSING
Part 1: 28 May 2018, 13:30–15:30

Closed book; two sides A4 of handwritten notes permitted

This exam consists of four questions (28 points). Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.

Question 1 (8 points)

Consider the following two probability density functions (PDFs) of the independent random variables X and Y:

$$f_X(x) = \begin{cases} \lambda_1 e^{-\lambda_1 x}\,, & \text{for} \quad x \geq 0 \\ 0\,, & \text{otherwise,} \end{cases}$$

$$f_Y(y) = \begin{cases} \lambda_2 e^{-\lambda_2 y}\,, & \text{for} \quad y \geq 0 \\ 0\,, & \text{otherwise.} \end{cases}$$

In this question we are interested in the random variable $Z = X + Y$.

a) Determine the distribution $f_Z(z)$ and show that it equals

$$f_Z(z) = \begin{cases} \dfrac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 z} - e^{-\lambda_2 z}\right)\,, & \text{for} \quad z \geq 0 \\ 0\,, & \text{otherwise.} \end{cases}$$

Hint: In the derivation you might want to use a partial fraction expansion.

b) Calculate the cumulative distribution function (CDF) $F_Z(z)$.

c) Determine the PDF $f_{Z|Z\geq 4}(z)$.

d) Calculate $E\left[Z|Z \geq 4\right]$.

Answer

(3 p) a) The MGF is

$$\phi_X(s) = E[e^{sX}] = \int_0^\infty e^{sx}\lambda_1 e^{-\lambda_1 x} dx = \int_0^\infty \lambda_1 e^{(s-\lambda_1)dx} = \frac{\lambda_1}{\lambda_1 - s}\,, \quad (\text{ROC: } s \leq \lambda_1)$$

Similarly,

$$\phi_Y(s) = \frac{\lambda_2}{\lambda_2 - s}\,, \quad (\text{ROC: } s \leq \lambda_2)$$

Then,

$$\phi_Z(s) = \phi_X(s)\phi_Y(s) = \frac{\lambda_1\lambda_2}{(\lambda_1 - s)(\lambda_2 - s)} = \frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}\left(\frac{1}{\lambda_1 - s} - \frac{1}{\lambda_2 - s}\right).$$

Using the already calculated MGF $\phi_X(s)$ we can then deduce that

$$f_Z(z) = \frac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 z} - e^{-\lambda_2 z}\right).$$

(2 p) b)

$$\begin{aligned}
F_Z(z) &= \int_0^z f_Z(u)du \\
&= \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\int_0^z \left(e^{-\lambda_1 u} - e^{-\lambda_2 u}\right) du \\
&= \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left[\frac{-1}{\lambda_1}e^{-\lambda_1 u} + \frac{1}{\lambda_2}e^{-\lambda_2 u}\right]_0^z \\
&= \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left(\frac{-1}{\lambda_1}e^{-\lambda_1 z} + \frac{1}{\lambda_2}e^{-\lambda_2 z} + \frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) \\
&= 1 + \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left(\frac{-1}{\lambda_1}e^{-\lambda_1 z} + \frac{1}{\lambda_2}e^{-\lambda_2 z}\right)
\end{aligned}$$

(1 p) c)

$$\begin{aligned}
P(Z \geq 4) &= 1 - P(Z \leq 4) \\
&= 1 - F_Z(4) \\
&= \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left(\frac{1}{\lambda_1}e^{-\lambda_1 4} - \frac{1}{\lambda_2}e^{-\lambda_2 4}\right)
\end{aligned}$$

$$f_{Z|Z\geq 4}(Z) = \begin{cases} \dfrac{\left(e^{-\lambda_1 z} - e^{-\lambda_2 z}\right)}{\frac{1}{\lambda_1}e^{-\lambda_1 4} - \frac{1}{\lambda_2}e^{-\lambda_2 4}}, & z \geq 4 \\ 0, & \text{otherwise.} \end{cases}$$

(2 p) d)

$$\begin{aligned}
\phi_{Z|Z\geq 4}(s) &= \int_4^\infty e^{sz}\frac{\left(e^{-\lambda_1 z} - e^{-\lambda_2 z}\right)}{\frac{1}{\lambda_1}e^{-\lambda_1 4} - \frac{1}{\lambda_2}e^{-\lambda_2 4}}dz \\
&= \int_4^\infty \frac{\left(e^{(s-\lambda_1)z} - e^{(s-\lambda_2)z}\right)}{\frac{1}{\lambda_1}e^{-\lambda_1 4} - \frac{1}{\lambda_2}e^{-\lambda_2 4}}dz \\
&= \frac{e^{4s}}{\frac{1}{\lambda_1}e^{-\lambda_1 4} - \frac{1}{\lambda_2}e^{-\lambda_2 4}}\left(\frac{e^{-\lambda_1 4}}{\lambda_1 - s} - \frac{e^{-\lambda_2 4}}{\lambda_2 - s}\right) \\
E\left[Z|Z\geq 4\right] &= \left.\frac{d\phi_{Z|Z\geq 4}(s)}{ds}\right|_{s=0} \\
&= \frac{4}{\frac{1}{\lambda_1}e^{-\lambda_1 4} - \frac{1}{\lambda_2}e^{-\lambda_2 4}}\left(\frac{e^{-\lambda_1 4}}{\lambda_1} - \frac{e^{-\lambda_2 4}}{\lambda_2}\right) + \frac{1}{\frac{1}{\lambda_1}e^{-\lambda_1 4} - \frac{1}{\lambda_2}e^{-\lambda_2 4}}\left(\frac{-e^{-\lambda_1 4}}{\lambda_1^2} + \frac{e^{-\lambda_2 4}}{\lambda_2^2}\right)
\end{aligned}$$

Question 2 (6 points)

Consider the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} \lambda e^{-\lambda x} & \text{for} \quad 0 \leq y \leq x \\ 0 & \text{otherwise.} \end{cases}$$

a) Calculate the PDFs $f_X(x)$ and $f_Y(y)$ and use these marginal PDFs to argue whether or not X and Y are dependent.

b) Calculate the MMSE estimator $\hat{Y} = E[Y|X]$.

c) Calculate the MMSE estimator $\hat{X} = E[X|Y]$.

Answer

(2 p) a) If $x \geq 0$ then

$$f_X(x) = \int_0^x \lambda e^{-\lambda x} dy = x\lambda e^{-\lambda x}$$

otherwise $f_X(x) = 0$. Similarly, if $y \geq 0$ then

$$f_Y(y) = \int_y^\infty \lambda e^{-\lambda x} dx = \left[-e^{-\lambda x}\right]_y^\infty = e^{-\lambda y}$$

otherwise $f_Y(y) = 0$.

RVs X and Y are dependent, as $f_{X,Y} \neq f_X f_Y$.

(2 p) b)

$$f_{Y|X}(y|x) = \begin{cases} f_{X,Y}/f_X = \frac{1}{x} & \text{for} \quad 0 \leq y \leq x \\ 0 & \text{otherwise.} \end{cases}$$

$$\hat{Y} = E[Y|X] = \int_0^x \frac{1}{x}\frac{1}{2} y dy = \frac{x}{2}.$$

(2 p) c) Calculate the MMSE estimator $\hat{X} = E[X|Y]$

$$f_{X|Y}(x|y) = \begin{cases} f_{X,Y}/f_Y = \lambda e^{-\lambda(x-y)} & \text{for} \quad y \leq x \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned} \hat{X} &= E[X|Y] = \int_y^\infty x\lambda e^{-\lambda(x-y)} dx \\ &= \lambda e^{\lambda y}\left(\left[-\frac{1}{\lambda} x e^{-\lambda x}\right]_y^\infty + \int_y^\infty \frac{1}{\lambda} e^{-\lambda x} dx\right) \\ &= y + \frac{1}{\lambda} \end{aligned}$$

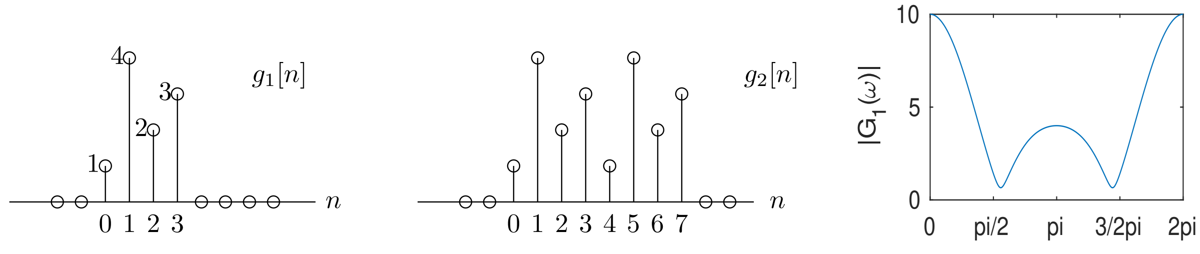
Or, using the MGF:

$$\phi_{X|Y}(s) = \frac{\lambda e^{ys}}{\lambda - s}$$

$$\left.\frac{d\frac{\lambda e^{ys}}{\lambda - s}}{ds}\right|_{s=0} = y + \frac{1}{\lambda}$$

Question 3 (8 points)

Consider the signal $g_1[n]$ in figure (a) and the signal $g_2[n]$ in figure (b). A plot of the DTFT $G_1(\omega)$ of $g_1[n]$ is shown in figure (c).



(a) (b) (c)

a) Derive that the DTFT of $g_2[n]$ is given by $G_2(\omega) = G_1(\omega) + e^{-4j\omega}G_1(\omega)$.

b) Take $N = 8$. What is the DFT of $g_1[n]$ $(n = 0, \cdots, N-1)$, expressed in terms of $G_1(\omega)$?

c) What changes if we take $N = 16$ or $N = 4$? Explain using a scetch of the spectrum.

d) What changes if we take the DFT of $g_2[n]$ (with $N = 8$)? Explain using a scetch.

e) Based on your result in item d), relate the DFT of $g_2[n]$ for $N = 8$ to the DFT of $g_1[n]$ for $N = 4$.

Answer

a)

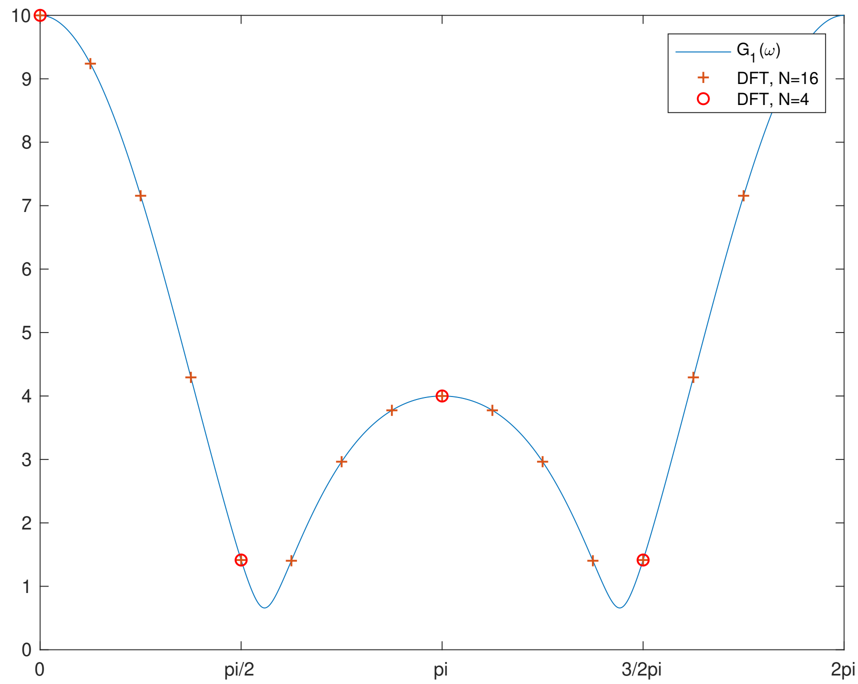
$$g_2[n] = g_1[n] + g_1[n-4] \quad \Rightarrow \quad G_2(\omega) = G_1(\omega) + e^{-4j\omega}G_1(\omega)$$

b)

$$G_1^{dft}[k] = G_1(\frac{2\pi}{8}k)\,, \qquad k = 0, \cdots, N-1\,.$$

These are samples of $G_1(\omega)$ for multiples of $\omega = 2\pi/N$.

c) Fewer, resp. more samples on the same curve $G_1(\omega)$. (zero-padding effect — interpolation in frequency)



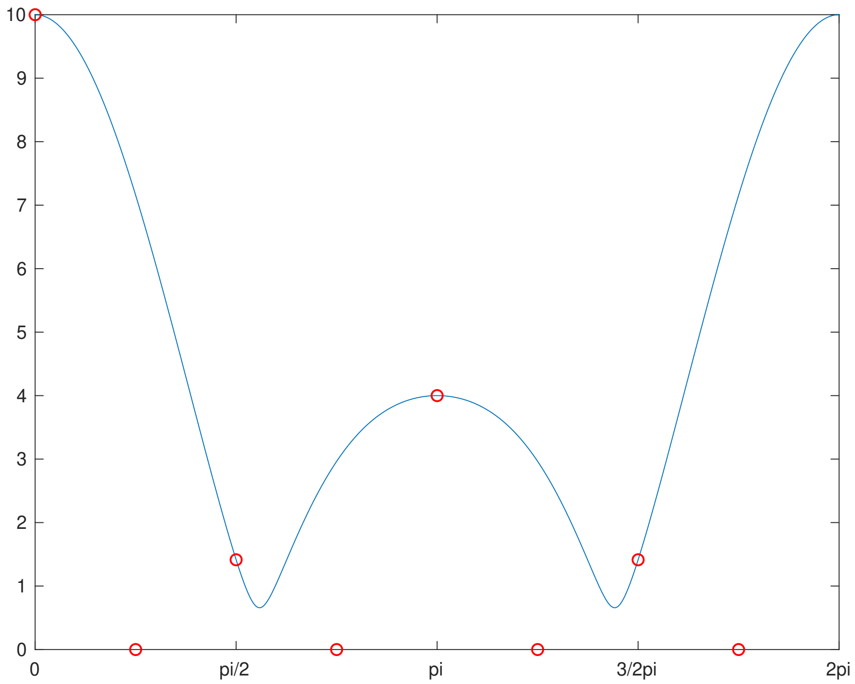
d)

$$G_2^{dft}[k] = G_1(\frac{2\pi}{8}k) + e^{-4j\frac{2\pi}{8}k}G_1(\frac{2\pi}{8}k) = G_1^{dft}[k] + (-1)^k G_1^{dft}[k]$$

For even sample indices k, this is equal to two times $G_1^{dft}[k]$ for $N = 8$ samples, i.e., equal to $N = 4$ samples. For odd sample indices, it is equal to zero.

Note: the spectrum $G_2^{dft}[k]$ for $N = 8$ is less detailed than the spectrum $G_1^{dft}[k]$ for $N = 8$. Thus, repeating a signal "to improve resolution" is not a good idea. With many repetitions, we converge to a line spectrum corresponding to the 4 samples of $G_1^{dft}[k]$ for $N = 4$.

e) Except for the factor two, we obtain the same spectrum as for the DFT of $g_1[n]$ with $N = 4$, interlaced with zeros.



Question 4 (6 points)

We are sampling an analog signal that has a bandwidth of $B = 5$ kHz. We can take samples during 2 seconds. We want to show the spectrum using the DFT.

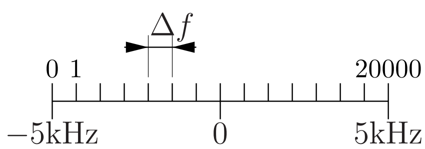
a) What is the minimal sample rate F_s needed to avoid aliasing?

b) At that rate, what is the resolution (in Hz) of the spectrum that we obtain? (Here, resolution is defined as the distance, in Hz, between two consecutive points in the spectrum.)

c) How can we improve the resolution by a factor 2? Give two options.

Answer

a) $F_s = 10$ kHz



b) $\Delta f = \frac{10k\ \text{Hz}}{20k\ \text{samples}} = 0.5$ Hz

c) Sample during 4 seconds, or zero pad to 4 seconds, using 20k zeros.

Note: doubling the sample rate does not help: the maximal frequency that can be observed will double but the resolution will remain 0.5 Hz.

Zero padding will not quite improve resolution but just interpolate the available samples.

Windowing will worsen resolution (but reduce the sidelobes so it can be helpful).