Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science
Section Circuits and Systems

Partial exam EE2S31 SIGNAL PROCESSING
Part 2: July 3, 2018

Closed book; two sides of one A4 of handwritten notes permitted

This exam consists of four questions (33 points). Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet.

Question 1 (8 points)

Given is the stochastic process $X(t) = A|t|$ with random variable A uniformly distributed in the (continuous) interval $[2, 4]$.

(1 p) (a) Sketch three different realizations of process $X(t)$.

3 p) (b) Calculate the cumulative distribution function (CDF) $F_X(t)(x)$, as well as the probability density function (PDF) $f_X(t)(x)$ for $t \neq 0$.

(1 p) (c) Calculate the expected value $E[X(t)]$.

(2 p) (d) Calculate the autocorrelation function $R(t, \tau)$.

(1 p) (e) Argue whether or not process $X(t)$ is stationary, and, whether or not process $X(t)$ is wide sense stationary (WSS).

Solution

(a)

(b) $F_{X(t)}(x) = P(X(t) \leq x) = P(A|t| \leq x) = P(A \leq \frac{x}{|t|}) = \int_2^{\frac{x}{|t|}} \frac{1}{2} da = \left[\frac{a}{2}\right]_2^{\frac{x}{|t|}} = \frac{1}{2}\frac{x}{|t|} - 1$ for $2|t| \leq x \leq 4|t|$. Altogether,

$$F_{X(t)}(x) = \begin{cases} \frac{1}{2}\frac{x}{|t|} - 1 & \text{for} \quad 2|t| \leq x \leq 4|t| \\ 0 & \text{otherwise.} \end{cases}$$

$$f_{X(t)}(x) = \frac{dF_{X(t)}(x)}{dx} = \begin{cases} \frac{1}{2}\frac{1}{|t|} & \text{for} \quad 2|t| \leq x \leq 4|t| \\ 0 & \text{otherwise.} \end{cases}$$

(c) $E[X(t)] = E[A]|t| = 3|t|$, or, using the calculated pdf:

$$E[X(t)] = \int_{-\infty}^{+\infty} x f_{X(t)}(x) = \int_{2|t|}^{4|t|} \frac{x}{2}\frac{1}{|t|} dx = \left[\frac{x^2}{4|t|}\right]_{2|t|}^{4|t|} = \frac{1}{4|t|}(16|t|^2 - 4|t|^2|) = 3|t|.$$

(d) $R(t, \tau) = E[A|t|A|t+\tau|] = E[A^2]|t||t+\tau| = \int_2^4 a^2 \frac{1}{2} da |t||t+\tau| = \frac{4^3-2^3}{6}|t||t+\tau| = \frac{28}{3}|t||t+\tau|$. Or, via the long route:

$$R(t, \tau) = E[A|t|A|t+\tau|] = \int_{2|t|}^{4|t|} \int_{2|t+\tau|}^{4|t+\tau|} x_1 x_2 f_{X(t),X(t+\tau)}(x_1, x_2) dx_2 dx_1$$

What is $f_{X(t),X(t+\tau)}(x_1, x_2)$?

$$f_{X(t),X(t+\tau)}(x_1, x_2) = f_{X(t+\tau)|X(t)}(x_2|x_1) f_{X(t)}(x_1) = \delta\left(x_1 - x_2 \frac{|t|}{|t+\tau|}\right) f_{X(t)}(x_1)$$

$$= \delta\left(x_1 - x_2 \frac{|t|}{|t+\tau|}\right) \frac{1}{2|t|}$$

(Given x_2, x_1 is completely determined.)

$$R(t,\tau) = E[A|t|A|t+\tau|] = \int_{2|t|}^{4|t|} \int_{2|t+\tau|}^{4|t+\tau|} x_1 x_2 \delta\left(x_1 - x_2 \frac{|t|}{|t+\tau|}\right) f_{X(t)}(x_1) dx_2 dx_1$$

$$= \int_{2|t|}^{4|t|} x_1^2 \frac{|t+\tau|}{|t|} \frac{1}{2|t|} dx_1 = \left[\frac{x_1^3}{6} \frac{|t+\tau|}{|t|^2}\right] \int_{2|t|}^{4|t|} = \frac{28}{3}|t||t+\tau|.$$

(e) Process $X(t)$ is not stationary, as the pdf ${}_{X(t)}(x)$ changes over time, and, $X(t)$ is not wide sense stationary (WSS) as both the expected value and autocorrelation function depend on time t.

Question 2 (8 points)

For this question you might want to make use of Table 1, included at the end of this exam.

Given is the time continuous WSS procecss $X(t)$ with autocorrelation function

$$R_X(\tau) = e^{-|\tau|100} + 4$$

and expected value $E[X(t)] = 2$. Process $X(t)$ is sampled with $f_s = 100$ samples per second, leading to the process $X[n]$.

(1 p) (a) Give the autocorrelation function $R_X[k]$ and expected value $E[X[n]]$ of the sampled process $X[n]$.

Given is a system with impulse response $h[n]$ and the abovementioned sampled process $X[n]$ as input. The output is denoted by $Y[n]$.

(1 p) (b) Assuming that $h[n]$ is of the form $h[n] = a^n u[n]$, with $u[n]$ the unit-step function, $|a| \leq 1$, and $E[Y[n]] = 8$, calculate constant a.

For the remaining part of this question, assume $R_X[k]$ is given by $R_X[k] = \left(\frac{1}{2}\right)^{|k|}$.

(2 p) (c) Give the magnitude response $|H(\phi)|$ that would completely decorrelate process $X[n]$.

Now assume $h[n] = \delta[n-3]$.

(2 p) (d) Calculate the cross-correlation $R_{XY}[k]$ between input and output.

(2 p) (e) Calculate the autocorrelation $R_Y[k]$ of the output.

Solution

(a) The process is WSS, so $E[X[k]] = E[X(t)]$ and $R_X[k] = R_X(kT_s) = e^{-|k|} + 4$.

(b) $E[Y] = E[X]\sum_{n=0}^{\infty} a^n = \frac{2}{1-a} = 8$, so $a = 3/4$.

(c) Decorrelated output, means $R_y[k] = c\delta[k]$, so $S_x(\phi) = c$, with c any constant.

$$S_x(\phi) = \frac{1-(1/2)^2}{1+(1/2)^2 - 2(1/2)\cos(2\pi\phi)} = \frac{3/4}{5/4 - \cos(2\pi\phi)}.$$

$$|H(\phi)| = \sqrt{\frac{S_y(\phi)}{S_X(\phi)}} = \sqrt{\frac{c}{\frac{1-(1/2)^2}{1+(1/2)^2-2(1/2)\cos(2\pi\phi)}}} = \sqrt{\frac{c(5/4-\cos(2\pi\phi))}{3/4}}.$$

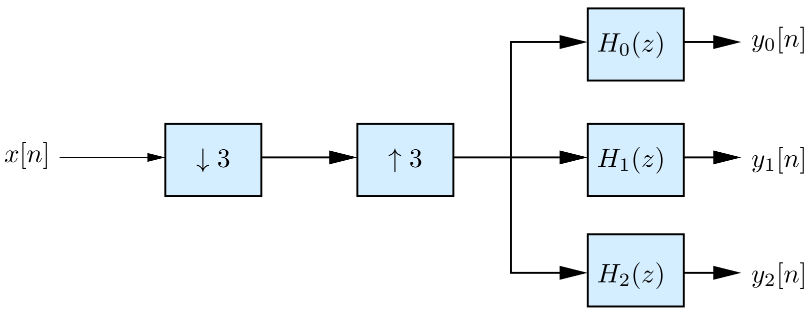
(d) $R_{XY}[k] = \sum_{j=0}^{\infty} \delta[j-3] \left(\frac{1}{2}\right)^{|k-j|} = \left(\frac{1}{2}\right)^{|k-3|}$

(e) A delay will not change autocorrelation of the input. The autocorrelation function therefore equals $R_Y[k] = R_X[k]$. Also, we can calculate

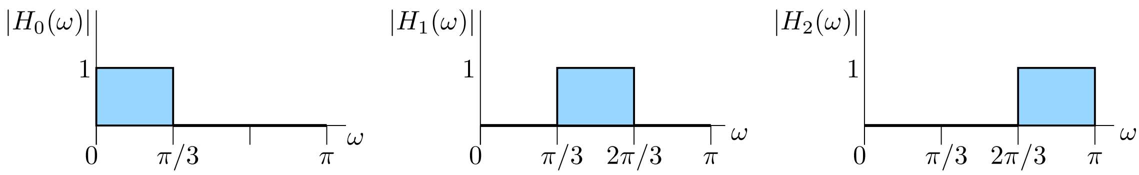
$$R_Y[k] = \sum_{i=-\infty}^{\infty} h[-i]R_{XY}[k-i] = \sum_{i=-\infty}^{\infty} \delta[-i+3]R_{XY}[k-i] = \sum_{i=-\infty}^{\infty} \delta[-i+3]\left(\frac{1}{2}\right)^{|k-3-i|} = \left(\frac{1}{2}\right)^{|k|}$$

Question 3 (6 points)

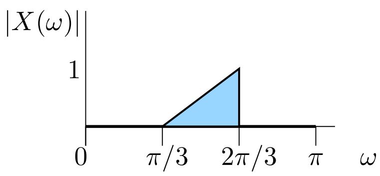
Consider the following multirate filter structure:



in which $H_0(z)$, $H_1(z)$ and $H_2(z)$ are a lowpass, bandpass, and highpas filter, respectively, as follows:



The real-valued input signal $x[n]$ has the following amplitude spectrum:

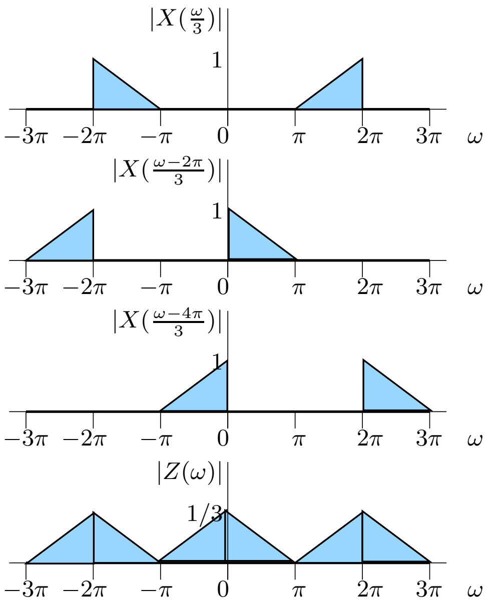


Draw the amplitude spectra of the outputs $y_0[n]$, $y_1[n]$, $y_2[n]$.

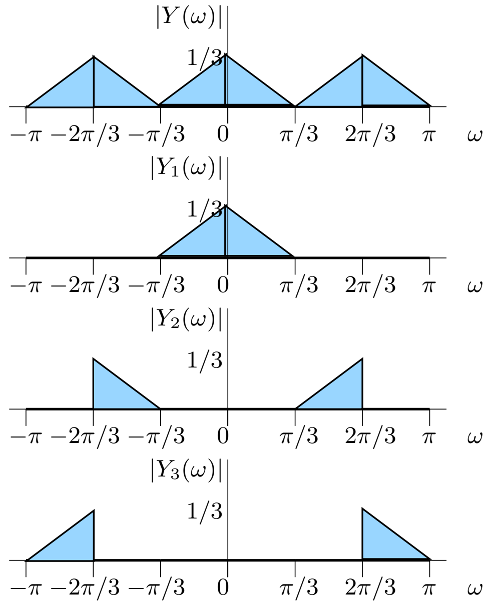
Solution

Because $x[n]$ is real-valued, it has a symmetric spectrum, and we will draw both sides.

After downsampling: $Z(\omega) = \frac{1}{3}\sum_{k=0}^{2} X(\frac{\omega - 2\pi k}{3})$. We draw a couple of periods:

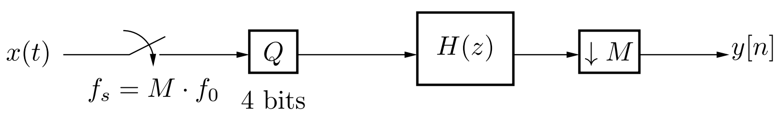


After upsampling: $Y(\omega) = Z(3\omega) = \frac{1}{3}\sum X(\omega - \frac{2}{3}\pi k)$



Question 4 (11 points)

An oversampling AD converter operates at $M = 256$ times the desired sampling rate $f_0 = 100$ kHz, en quantizes samples at 4 bits. Digitally, the sample rate is reduced by a factor $M = 256$. See figure:



The filter is given by $H(z) = 1 + z^{-1} + \cdots + z^{-(M-1)}$.

a Draw the amplitude spectrum of $H(z)$ (clearly mark the frequency axis).

  In this plot, what is the location (value of ω) of the first zero of the transfer function?

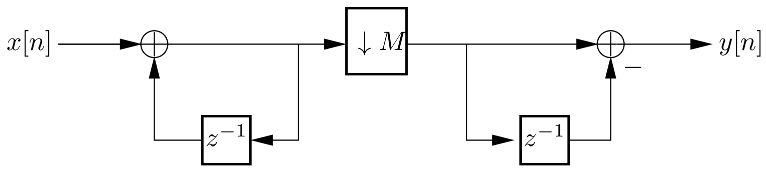
b What is the role of this filter?

  If it was an ideal filter, what would be its specification?

c What is the effect of this filter on the quantization noise?

d Assume that the quantization noise is white. How many bits accuracy do you expect at the output? (Why?)

e An efficient implementation of the filter, in combination with the downsampler, is as follows:



  Prove that this implementation indeed results in the desired response.

f Give an advantage and a disadvantage of this filter, compared to a more general Mth order FIR filter.

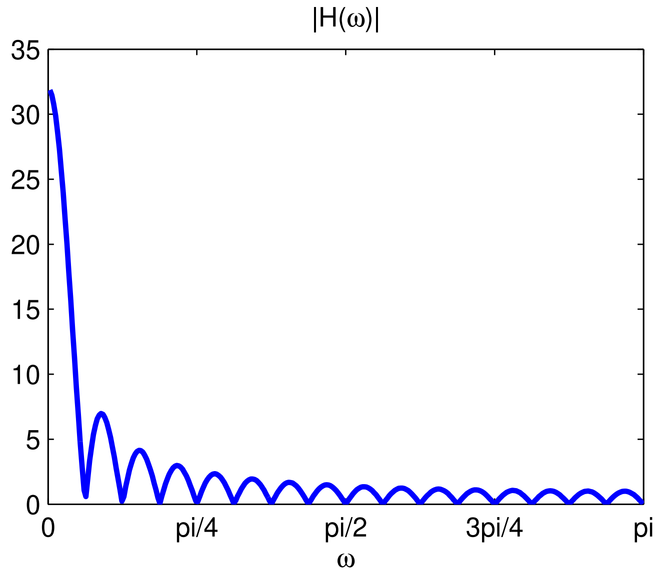
g To obtain a better filter response, we now use two of such filters $H(z)$ in cascade. Draw a block diagram of an *efficient* implementation (in combination with the downsampler).

Solution

a Recognize the usual digital sinc function (Dirichlet function). We can write:

$$H(z) = \frac{1 - z^{-M}}{1 - z^{-1}}$$

  $H(z)$ has M zeros, at $z = e^{j\frac{2\pi}{M}k}$ $(k = 0, \cdots, M-1)$, and a pole at $z = 1$ which will cancel the corresponding zero. The first zero is at $\omega = 2\pi/M$, which corresponds to 100 kHz.



(Drawn for $M = 32$)

b A lowpass anti-aliasing filter for the downsampling. Ideally, it cuts off at $\omega = \pi/M$, the given non-ideal filter has its 3dB point at this frequency.

c This noise is also filtered, and we will lose a lot of noise (ideally, a fraction $1/M$ is retained).

Alternatively, we can say that the noise is being averaged ($H(z)$ sums 256 subsequent samples), this will reduce the variance of the noise by a factor M (assuming the noise is white).

(Note that in the given situation this assumption is not really valid. Normally, in such ADCs the difference between two subsequent samples is digitized. That differene has mean almost equal to zero, so that the quantization noise is much more independent. See: Sigma-Delta ADC.)

d Initially, the variance of the quantization noise was $\frac{q^2}{12} = \frac{2^{-2B}}{12}$, where $B = 4$ bits.

Due to the averaging in the filter, the variance is reduced by a factor M: $\frac{2^{-2B}}{12M}$. If we equate this to $\frac{2^{-2B'}}{12}$, then the new number of bits becomes

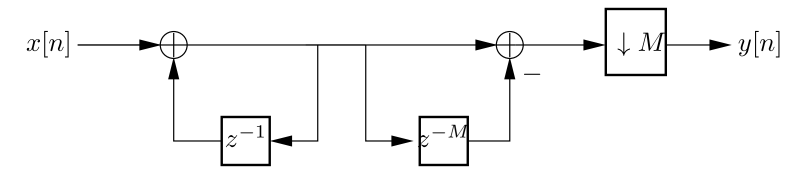
$$\frac{2^{-2B'}}{12} = \frac{2^{-2B}}{12M} \quad \Rightarrow \quad -2B' = -2B - \log_2 M \quad \Rightarrow \quad B' = B + \frac{1}{2}\log_2 M = 4 + 4 = 8$$

(A Sigma-Delta ADC would be much better at this: $4 + 12 = 16$ bits, because in that case the quantization noise in the passband is strongly filtered.)

e

$$H(z) = \frac{1 - z^{-M}}{1 - z^{-1}}$$

An initial realization could be:



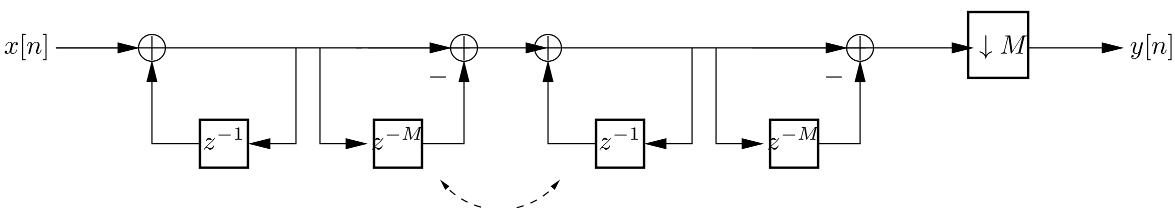
We can bring the downsampler more to the front, this will replace z^{-M} by z^{-1}. (Cf. "noble identities")

(A derivation of $H(z)$ using the formulas for upsampling/downsampling is more tricky, e.g. because this is not an LTI system: an expression like $Y(z) = H(z)X(z)$ is not valid.)

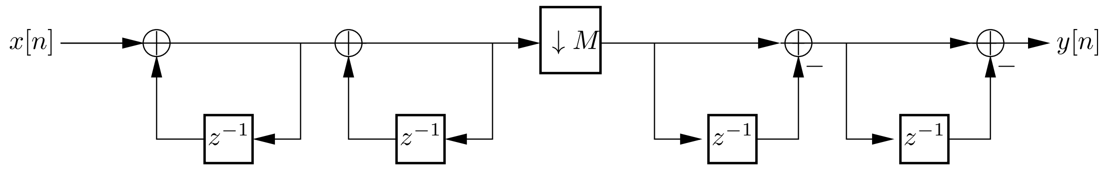
f +: can be implemented without multipliers and using only 2 delays;

−: not a very good lowpass filter (resulting in aliasing), possibly instable (pole at the unit circle, it is being canceled later by a zero but care must be taken in the realization).

g An initial realization would be as follows:



Since convolution is commutative, we can swap the feedforward part with the feedback part. Next, we can move the downsampler to the left as before:



This filter has a better lowpass characteristic.

Discrete Time function	Discrete Time Fourier Transform
$\delta[n] = \delta_n$	1
1	$\delta(\phi)$
$\delta[n - n_0] = \delta_{n-n_0}$	$e^{-j2\pi\phi n_0}$
$u[n]$	$\dfrac{1}{1 - e^{-j2\pi\phi}} + \dfrac{1}{2}\sum_{k=-\infty}^{\infty}\delta(\phi + k)$
$e^{j2\pi\phi_0 n}$	$\sum_{k=-\infty}^{\infty}\delta(\phi - \phi_0 - k)$
$\cos 2\pi\phi_0 n$	$\frac{1}{2}\delta(\phi - \phi_0) + \frac{1}{2}\delta(\phi + \phi_0)$
$\sin 2\pi\phi_0 n$	$\frac{1}{2j}\delta(\phi - \phi_0) - \frac{1}{2j}\delta(\phi + \phi_0)$
$a^n u[n]$	$\dfrac{1}{1 - ae^{-j2\pi\phi}}$
$a^{\lvert n \rvert}$	$\dfrac{1 - a^2}{1 + a^2 - 2a\cos 2\pi\phi}$
g_{n-n_0}	$G(\phi)e^{-j2\pi\phi n_0}$
$g_n e^{j2\pi\phi_0 n}$	$G(\phi - \phi_0)$
g_{-n}	$G^*(\phi)$
$\sum_{k=-\infty}^{\infty} h_k g_{n-k}$	$G(\phi)H(\phi)$
$g_n h_n$	$\int_{-1/2}^{1/2} H(\phi')G(\phi - \phi')\, d\phi'$

Table 1. Discrete-Time Fourier transform pairs and properties. (from Signal Processing Supplement of "probability and stochastic processes" by Yates and Goodman).