Delft University of Technology
Faculty of Electrical Engineering, Mathematics, and Computer Science

EE2S1 SIGNALS AND SYSTEMS

Part 2 exam, 5 November 2025, 9:00–11:00

Closed book; one A4 (two sides) of handwritten notes permitted. No other tools except a basic pocket calculator permitted.

This exam consists of five questions (23 points). Answer in Dutch or English. Make clear in your answer how you reach the final result; the road to the answer is very important.

Question 1 (7 points)

(a) Given the signal $x[n] = [\cdots, 0, \boxed{1}, 3, 0, 0, \cdots]$.

Determine $y[n] = x[n] * x[-n]$ using the convolution sum (in time-domain).

(b) Also determine $y[n]$ in (a) via the z-transform (do you obtain the same result?).

(c) Given $x[n] = (n-1)\,u[n]$. Determine $X(z)$, also specify the ROC.

(d) Given $X(z) = \dfrac{4z}{(z-1)(z+0.25)}$, $\qquad$ ROC $= \{0.25 < |z| < 1\}$.

Determine $x[n]$ using the inverse z-transform.

(e) Let $x[n] = u[n+2] - u[n-3]$. Determine the DTFT $X(e^{j\omega})$.

(f) Suppose the DTFT of a signal $x[n]$ is $X(e^{j\omega})$. What is the DTFT of $\cos(3n) \cdot x[n]$?

Solution

1p (a) Define $r[n] = x[-n] = [\cdots, 0, 3, \boxed{1}, \cdots]$. Using $y[n] = x[n] * r[n] = \sum_{k=-\infty}^{\infty} x[k]r[n-k]$, we find

$$
\begin{array}{rcl}
(k=0)\ 1 \cdot r[n] & = & [\cdots,\ 0,\ 3\ \ \boxed{1},\ \ 0,\ 0,\ \cdots] \\
(k=1)\ 3 \cdot r[n-1] & = & [\cdots,\ 0,\ 0,\ \boxed{9},\ \ 3,\ 0,\ \cdots] \\
\hline
y[n] & = & [\cdots,\ 0,\ 3,\ \boxed{10},\ 3,\ 0,\ \cdots]
\end{array}
$$

1p (b) Since $X(z) = 1 + 3z^{-1}$ and the z-transform of $x[-n]$ is $X(z^{-1})$, we find

$$Y(z) = X(z) \cdot X(z^{-1}) = (1 + 3z^{-1})(1 + 3z) = 3z + 10 + 3z^{-1}$$

Hence $y[n] = [\cdots, 0, 3, \boxed{10}, 3, 0, \cdots]$.

1p (c) Using the table,

$$X(z) = \frac{z^{-1}}{(1-z^{-1})^2} - \frac{1}{1-z^{-1}} = \frac{z^{-1} - (1-z^{-1})}{(1-z^{-1})^2} = \frac{2z^{-1} - 1}{(1-z^{-1})^2}$$

ROC: $\{|z| > 1\}$.

2p (d) First write as function of z^{-1} and do a partial fraction expansion:

$$X(z) = \frac{4z^{-1}}{(1-z^{-1})(1+0.25z^{-1})} = \frac{16/5}{1-z^{-1}} - \frac{16/5}{1+0.25z^{-1}}$$

(Alternative, keep as function of z.) Check the ROC: the first term corresponds to an anticausal sequence, the second is causal. Hence rewrite

$$X(z) = -\frac{16/5z}{1-z} - \frac{16/5}{1+0.25z^{-1}}$$
$$\Rightarrow\ x[n] = -\frac{16}{5}u[-n-1] - \frac{16}{5}\left(-\frac{1}{4}\right)^n u[n]$$

Depending on your PFE method, many alternative but equivalent answers are possible here. E.g.,

$$x[n] = -\frac{16}{5}u[-n] + \frac{4}{5}\left(-\frac{1}{4}\right)^{n-1} u[n-1]$$

1p (e) Write $x[n] = [\cdots, 0, 1, 1, \boxed{1}, 1, 1, 0, \cdots]$, so that

$$X(e^{j\omega}) = e^{j2\omega} + e^{j\omega} + 1 + e^{-j\omega} + e^{-j2\omega} = 1 + 2\cos(\omega) + 2\cos(2\omega)$$

This can also be obtained from the z-transform of the original function, but notice the pole-zero cancellation, which has to be taken into account before substituting $z = e^{j\omega}$, otherwise the ROC doesn't include the unit circle.

Alternatively, the z-transform of a shifted pulse gives

$$X(e^{j\omega}) = \frac{\sin(\frac{5}{2}\omega)}{\sin(\frac{1}{2}\omega)}$$

which is actually the same.

1p (f) Use the modulation property of the DTFT:

$$Y(\omega) = \frac{1}{2}\left(X(\omega-3) + X(\omega+3)\right)$$

Question 2 (3 points)

A causal system is specified by the transfer function

$$H(z) = \frac{z^2+1}{(z-0.9)(z+0.9)}\,.$$

(a) Determine all poles and zeros of the system and draw a pole-zero plot.

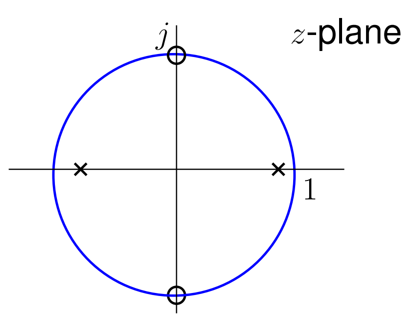
(b) What is the ROC?

(c) Is this a stable system? (Motivate.)

(d) Sketch the amplitude spectrum $|H(e^{j\omega})|$, also indicate values on the frequency axis.

Solution

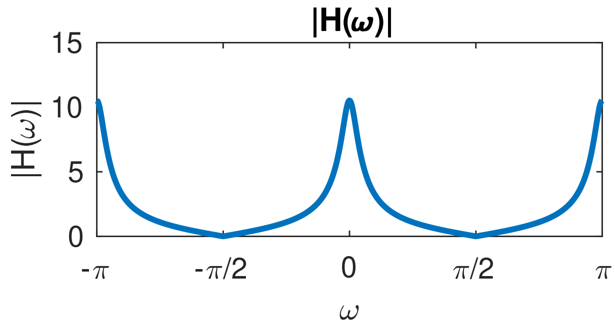
1p (a) Poles: $z = 0.9$, $z = -0.9$; zeros: $z = \pm j$.



0.5p (b) Causal, hence ROC: $\{|z| > 0.9\}$.

0.5p (c) Unit circle in ROC: stable. (Or: causal and no poles on or outside the unit circle.)

1p (d) Use fasors:

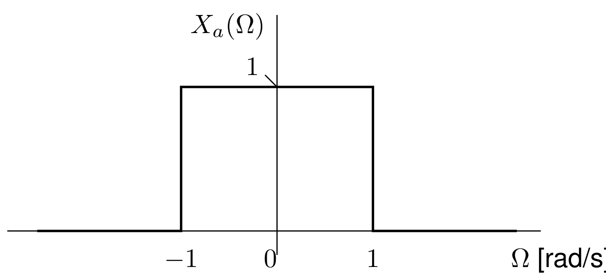


Peaks at $\omega = 0, \pi$. The peaks have equal heights, due to symmetries. At the peaks the function is flat (hard to see in the matlab plot, but you can indicate this more clearly in the sketch). Evaluate $|H(\omega = 0)| = \frac{2}{0.1 \cdot 1.9} = 10.530$.

Zeros at $\omega = \pm\pi/2$. At the zeros, the function is not flat (similar to $|\cos(\omega)|$ at a zero crossing of $\cos(\omega)$).

Question 3 (5 points)

An analog signal $x_a(t)$ has Fourier Transform $X_a(\Omega) = u(\Omega + 1) - u(\Omega - 1)$, as shown in the figure.



(a) Determine $x_a(t)$.

(b) What is the largest value of the sample period T_s that would not cause aliasing when sampling $x_a(t)$?

(c) For that T_s, let the resulting sampled signal be $x[n]$. Sketch the spectrum $X(e^{j\omega})$ (carefully label the axes).

(d) Let $y_a(t) = (x_a(t))^2$. Give an expression for $Y_a(\Omega)$ in terms of $X_a(\Omega)$.

(e) If we sample $y_a(t)$ at the same period T_s as before in (c), would aliasing occur? (Motivate your answer using a sketch.)

Solution

1p (a) Using the Inverse Fourier Transform,

$$x_a(t) = \frac{1}{2\pi} \int_{-1}^{1} e^{j\Omega t} \mathrm{d}\Omega = \frac{1}{2\pi} \left[\frac{1}{jt} e^{j\Omega t} \right]_{-1}^{1} = \frac{1}{2\pi} \frac{e^{jt} - e^{-jt}}{jt} = \frac{\sin(t)}{\pi t}$$

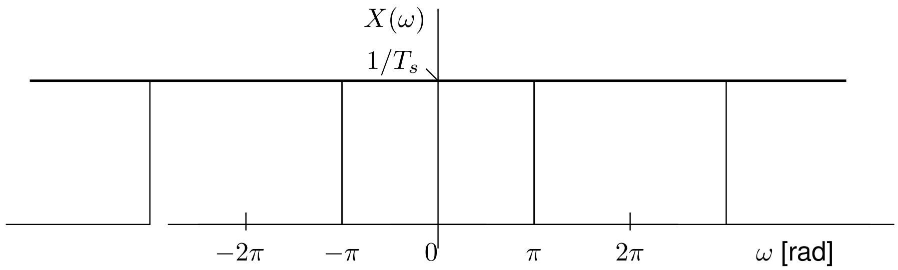
1p (b) The bandwidth is $B = 1$ [rad/s]. Since $\Omega_s \geq 2B = 2$, and $\Omega_s = 2\pi F_s$ and $F_s = 1/T_s$, then

$$T_s = \frac{1}{F_s} = \frac{2\pi}{\Omega_s} \leq \frac{2\pi}{2B} = \pi$$

1p (c)

$$X(e^{j\omega}) = \frac{1}{T_s} \sum_k X_a(\Omega - k\Omega_s)\,, \qquad \omega = \Omega T_s$$

The spectrum repeats itself with period 2π, corresponding to $2B$. The height is $1/T_s$.

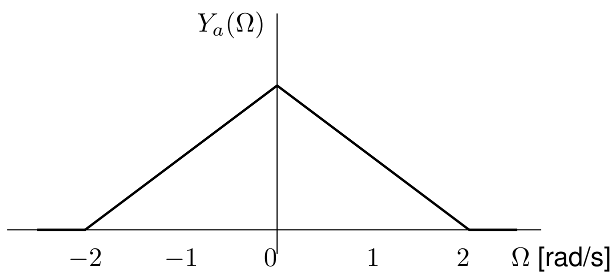


The spectrum is flat, so the corresponding signal is $x[n] = \frac{1}{pi}\delta[n]$. This is consistent with the result in (a) if we evaluate $x[n] = x_a(nT_s)$.

1p (d) Multiplication in time gives convolution in frequency:

$$Y_a(\Omega) = \frac{1}{2\pi} X_a(\Omega) * X_a(\Omega)$$

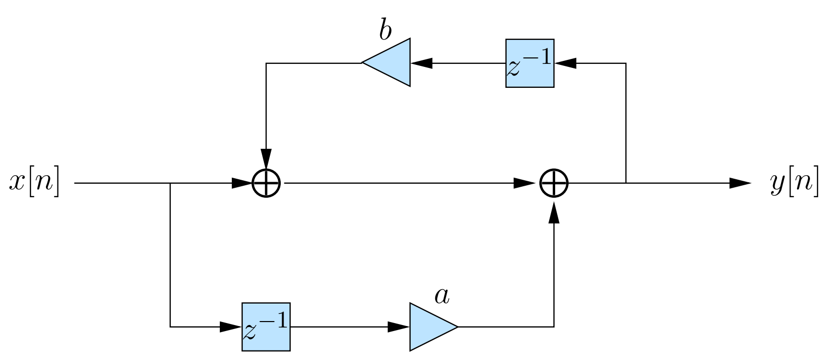
1p (e) Due to the convolution, the spectrum $Y_a(\Omega)$ is twice the width of that of $X_a(\Omega)$; in fact it will be triangular, see figure.



Since the Nyquist condition is not satisfied, aliasing will occur.

Question 4 (3 points)

(a) Determine the transfer function $H(z)$ of the following realization:



(b) Is this a minimal realization? (Why?)

(c) Draw the "Direct form no. II" realization and also specify the coefficients.

Solution

1.5p (a) Insert additional variables: call $P(z)$ the output of the multiplier a, and $Q(z)$ the output of the multiplier b.

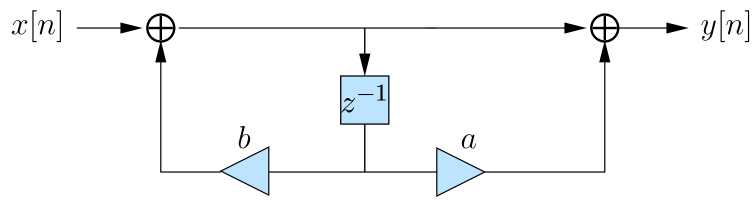
$$\begin{aligned} P(z) &= az^{-1}X(z) \\ Q(z) &= bz^{-1}Y(z) \\ Y(z) &= X(z) + P(z) + Q(z) \end{aligned}$$

so that $Y(z) = X(z) + P(z) + Q(z) = X(z) + az^{-1}X(z) + bz^{-1}Y(z)$, and

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + az^{-1}}{1 - bz^{-1}}$$

0.5p (b) Not minimal: the number of delay elements is 2 but the filter is first order.

1p (c) Direct form II realization:



Question 5 (5 points)

A "template" second order analog lowpass filter (normalized Butterworth filter) is given by

$$H_a(s) = \frac{1}{s^2 + \sqrt{2}\,s + 1} .$$

We will design a second-order digital *high*pass filter $G(z)$ with 3 dB cut-off frequency $\omega_c = \frac{3}{4}\pi$. For this, we first design a suitable highpass filter $G_a(s)$ in the analog domain, and then apply the bilinear transformation.

(a) For the template $H_a(s)$, demonstrate that the corresponding magnitude squared frequency response is

$$|H_a(j\Omega)|^2 = \frac{1}{1 + \Omega^4} ,$$

showing that $H_a(s)$ is, indeed, a second-order Butterworth lowpass filter.

(b) For our design, what has to be the cut-off frequency in the analog frequency domain?

(c) Which frequency transformation do you apply to the template to arrive at an analog highpass filter with the desired cut-off frequency? What is the resulting $|G_a(j\Omega)|^2$?

(d) For this filter, at what frequency do we have a damping of 10 dB? What will be the corresponding frequency in digital domain?

(e) What is the resulting analog-domain highpass filter transfer function $G_a(s)$?

(f) What is the resulting $G(z)$?

Solution

0.5p (a) Use $s = j\Omega$:

$$H_a(j\Omega) = \frac{1}{(j\Omega)^2 + \sqrt{2}(j\Omega) + 1} = \frac{1}{1 - \Omega^2 + j\sqrt{2}\Omega}$$
$$|H_a(j\Omega)|^2 = \frac{1}{1 - \Omega^2 + j\sqrt{2}\Omega}\frac{1}{1 - \Omega^2 - j\sqrt{2}\Omega}$$
$$= \frac{1}{(1-\Omega^2)^2 - 2\Omega^2} \quad = \frac{1}{1+\Omega^4}$$

1p (b) Using the bilinear transformation gives cut-off frequency: $\Omega_c = \tan(\omega_c/2) = 2.4142$ rad/s.

1p (c) Frequency transformation (lowpass to highpass):

$$\Omega \to \frac{\Omega_c}{\Omega}, \qquad s \to \frac{\Omega_c}{s}$$

For our design, this gives $\Omega \to \frac{2.4142}{\Omega}$ and $s \to \frac{2.4142}{s}$.

$$|G_a(j\Omega)|^2 = \frac{1}{1 + \left(\frac{2.4142}{\Omega}\right)^4}.$$

1p (d) We need to determine Ω_s:

$$|G_a(j\Omega_s)|^2 = \frac{1}{1 + \left(\frac{2.4142}{\Omega_s}\right)^4} = 10^{-10/10} = 0.1\,.$$

This gives

$$\left(\frac{2.4142}{\Omega_s}\right)^4 = 10 - 1 = 9$$
$$\frac{2.4142}{\Omega_s} = \sqrt{3}$$
$$\Omega_s = 1.39\ [\text{rad/s}]\,.$$

In digital domain (based on the bilinear transform) this corresponds to

$$\omega_s = 2\tan^{-1}(\Omega_s) = 1.90\ [\text{rad}]\,.$$

0.5p (e) From (c), the analog domain highpass filter transfer function is

$$G_a(s) = \frac{1}{(\frac{2.4142}{s})^2 + \sqrt{2}\frac{2.4142}{s} + 1} = \frac{s^2}{s^2 + 3.4142s + 5.8284}.$$

1p (f) The corresponding digital transfer function follows from inserting the bilinar transform:

$$s \to \frac{1 - z^{-1}}{1 + z^{-1}}.$$

This results in

$$H(z) = \frac{(\frac{1-z^{-1}}{1+z^{-1}})^2}{(\frac{1-z^{-1}}{1+z^{-1}})^2 + 3.4142(\frac{1-z^{-1}}{1+z^{-1}}) + 5.8284} = \cdots = \frac{z^2 - 2z + 1}{10.2436z^2 + 9.6568z + 3.4142}.$$